

26th INTERNATIONAL MATHEMATICAL OLYMPIAD

Finland, July 1-11, 1985

Introduction

38 countries participated, Luxembourg, Mexico and Venezuela were absent. China (2), Iceland (2) and Iran (1) appeared for the first time and India sent an observer. The other teams with less than the maximum 6 competitors were Italy (5), Kuwait (5), Spain (4) and Tunis (4).

Thus there were 209 competitors from 38 different countries. About a dozen were girls.

The Choice of Problems

On June 29, two days before the teams arrived, the leaders met in Heinola as a Jury in order to decide the six problems of the competition. Several weeks before, each country had sent up to 5 problem-proposals to Finland. From these a selection of 6 first choices and further second choices had been made by the Finnish problem-selector for the Jury to consider.

In recent years some countries have felt that the range of mathematical knowledge required was too limited and the Finnish problem-selector had asked for a wider range of proposals. Three of the five problems submitted by G.B. involved knowledge of coordinate geometry applied to 1) plane similarity transformations, 2) the constant sum of focal distances in an ellipse and 3) two versions of a partitioning problem, one couched in the language of probability. None of these was selected for the consideration of the Jury. One 'traditional' G.B. problem was a second choice and the other was later produced when other geometry problems had been discarded, and finally selected as Question 1.

After the problem-papers had been decided, the problem-selector issued a pamphlet containing all proposals received with their solutions. It is evident from these proposals and discussion in the Jury that although some are in favour of reducing school geometry, particularly 3-D, and a few for introducing items in newer school syllabuses, most are content with the status quo.

Leaders come to the IMO having spent time (in some cases considerable time) training their teams and they are not going to vote for problems which seem to them outside the unwritten IMO syllabus. Should perhaps notice be given by host countries that proposals involving say complex numbers, coordinate geometry applied to curves and surfaces other than the circle and sphere, probability, etc. etc., will be accepted, or is this not worth while because such proposals will be voted down by the Jury?

The tradition is that the first and last of the 3 questions set on each of the two days should be 'easy' and 'difficult' respectively. Q3 proved to be the most difficult but it was fair in the sense that familiarity with binomial coefficients is to be expected and the difficulty lay in constructing a proof.

On the other hand Q6 involved the idea of nested intervals - two sequences converging to the same limit from above and below. It was a very nice question but comparatively easy for those whose approach to analysis starts after school with such ideas rather than as in G.B. with 'engineering' calculus in school. The U.S. leader had done no preparation in analysis of this sort in his 3-week training session and attributed the U.S. success (40/42) in this question to the fact that many U.S. universities hold week-end conferences for promising youngsters and a favourite lecture is on first year analysis, strange functions, etc.

Many of the leaders are university lecturers or professors in pure mathematics and much against 'calculus' being done in schools where the intricacies of limiting processes cannot be 'properly' explained. No doubt they expected that such ideas might be known or discovered by the abler pupils in their teams, as I did (cf. Area under a curve lying between the sum of upper and lower rectangular areas).

Of the two geometry questions Q1 and Q5, Q1 could be solved (inelegantly no doubt) by almost any sensible approach (e.g. using trigonometry). Q5 was chosen after an excellent 3-D polyhedron problem had been rejected and a substitute on vectors rejected as too easy. Eventually Q5 with its unnecessarily hard official solution was selected. (In fact both Q1 and Q5 were solved by G.B. competitors using no more than angle properties of quadrilateral and circle). From table 2, it can be seen that some leading countries find geometry difficult.

Q2 (combinatorics) and Q4 (prime factorisation and the pigeonhole principle) were both accepted without controversy.

The Competition

On July 4 and 5 the competition ($4\frac{1}{2}$ hours each morning) took place in Joutsa where the deputies and teams had been housed since July 1. On July 5 the leaders left Heinola for Joutsa and were no longer held incommunicado from their teams. Deputies and leaders marked and submitted the scripts for each question to the Finnish co-ordinators for that question. The size of IMO is now such that each question has two pairs of co-ordinators and two days passed before all co-ordination was ended. Table 1 shows the British competitors, their marks and prizes and Table 2 shows the marks per question of those 14 countries whose total marks exceeded 100.

The remaining countries are those named in paragraph 1 and the following: Brazil, Israel, Austria, Cuba, Netherlands, Greece, Yugoslavia, Sweden, Mongolia, Belgium, Morocco, Colombia, Turkey, Algeria, Norway, Cyprus and Finland.

Table 1

		Question	1	2	3	4	5	6	Total	Prize
C. Kilgour	King's School, Gloucester		7	0	0	2	2	1	12	
J. Longley	Portsmouth Grammar School		7	7	0	2	0	0	16	III
M. Moore	Manchester Grammar School		7	7	3	0	0	0	17	III
M. Richards	Millfield School		7	7	1	7	7	3	32	II
A. Selby	City of London School		7	7	0	2	0	1	17	III
I. Stark	Winchester College		7	7	2	4	7	0	27	II
			42	35	6	17	16	5	121	
Average mark for a team of six			25	22	5	15	11	12	90	

George Megyesi of Atlantic College, Glamorgan was chosen for the Hungarian team. He shared first prize in the British Mathematical Olympiad with Matthew Richards and got one of the two perfect scores (Ian Stark got the other) in our multiple-choice National Mathematics Competition. His mark in IMO was 38. Only 2 competitors (another Hungarian and a Romanian) did better, each getting the maximum 42.

Table 2

		Question	1	2	3	4	5	6	Total	Prizes		
										I	II	III
1	Romania		42	42	16	35	32	34	201	3	3	
2	United States		28	38	34	30	10	40	180	2	4	
3	Hungary		42	31	17	30	23	25	168	2	2	2
4	Bulgaria		42	35	10	22	33	23	165	2	3	
5	Vietnam		42	35	2	25	28	12	144	1	4	
6	Soviet Union		28	42	21	14	15	20	140	1	2	2
7	West Germany		25	41	16	26	8	23	139	1	1	4
8	East Germany		36	35	11	19	10	25	136		3	3
9	France		15	42	1	24	13	30	125		2	3
10	Great Britain		42	35	6	17	16	5	121		2	3
11	Australia		34	33	7	19	10	14	117	1	1	2
12=	Canada		30	42	2	15	9	7	105		1	4
12=	Czechoslovakia		22	31	1	15	14	22	105		3	1
14	Poland		33	35	1	4	11	17	101		1	4

The remaining 24 countries scored less than 100 marks,
Belgium (total score 60) had a first prize winner.
Soviet Union's first prize winner was a girl.

The British Team

Their overall performance was disappointing. Matthew Richards was unlucky not to get a first prize. First, second, third and no prizes have commonly been given in the ratio 1:2:3:6. This year only 14 first prizes were given and Richards had the highest mark not to get a first prize.

There were several ingenious attempts and David Monk spent some time trying to 'rescue' solutions which proved to be too unsound for the co-ordinators to allow few if any marks.

We noticed rather more obscurity in presentation than in previous years. Some of the team obviously had an off-day and did not do themselves justice. Two British solutions were put up for special prizes but were not considered 'special' or striking enough. No special prizes were given this year.

The six British competitors were a very pleasant lot. They got on well with each other and with their English-speaking rivals. They enjoyed themselves and did not let their disappointments get them down.

Non-Mathematical Matters

We are very grateful to our Finnish hosts, not only for the excellent organisation of the Competition, but for the programme of sightseeing, informal games, football, tennis, boating, swimming, saunas, picnics and concerts, all contributing opportunities for 'friendship between the young of 38 nations'. Teams had young Finnish students as attachés and our attachée was very helpful indeed looking after our general interests, keeping us up to the mark and overcoming language difficulties.

On the last evening various countries contributed to an informal variety show. Our team sang two Swan-songs - 'The English are Best' and 'Mud, mud, Glorious Mud.'

The Future

At the last meeting of the Jury, John Hersee, Secretary of the IMO Site Committee, told us that future host countries were - 1986, Poland: 1987, Cuba: 1988, Australia. Sweden and West Germany were among those to follow. The difficulties in wide mathematical disparity and in the organising of ever-increasing numbers were discussed. Surprisingly, the increasing cost to future host countries was not thought too heavy by most of the five leaders concerned above. They seemed confident that governmental, industrial and academic support will be readily forthcoming.

Robert Lyness.



FIRST DAY

Joutsa July 4, 1985

The name of the country proposing the question has been added at the end of each question. On the papers given to competitors these names were not given.

1. A circle has centre on the side AB of the cyclic quadrilateral $ABCD$.

The other three sides are tangent to the circle. Prove that $AD + BC = AB$.

Great Britain.

2. Let n and k be given relatively prime natural numbers, $0 < k < n$.

Each number in the set $M = \{1, 2, \dots, n-1\}$ is coloured either blue or white. It is given that

(i) for each $i \in M$, both i and $n-i$ have the same colour, and

(ii) for each $i \in M$, $i \neq k$, both i and $|i - k|$ have the same colour.

Prove that all numbers in M must have the same colour.

Australia.

3. For any polynomial $P(x) = a_0 + a_1x + \dots + a_kx^k$ with integer coefficients, the number of coefficients which are odd is denoted by $w(P)$. For $i = 0, 1, 2, \dots$ let $Q_i(x) = (1+x)^i$. Prove that if i_1, i_2, \dots, i_n are integers such that $0 \leq i_1 < i_2 < \dots < i_n$, then

$$w(Q_{i_1} + Q_{i_2} + \dots + Q_{i_n}) \geq w(Q_{i_1}).$$

Netherlands.

Time allowed: $4\frac{1}{2}$ hours

Each problem is worth 7 points.



SECOND DAY

Joutsa July 5, 1985

4. Given a set M of 1985 distinct positive integers, none of which has a prime divisor greater than 26. Prove that M contains at least one subset of four distinct elements whose product is the fourth power of an integer.

Mongolia.

5. A circle with centre O passes through the vertices A and C of triangle ABC , and intersects the segments AB and BC again at distinct points K and N , respectively. The circumscribed circles of the triangles ABC and KBN intersect at exactly two distinct points B and M . Prove that angle OMB is a right angle.

Soviet Union.

6. For every real number x_1 , construct the sequence x_1, x_2, \dots by setting

$$x_{n+1} = x_n \cdot \left(x_n + \frac{1}{n} \right)$$

for each $n \geq 1$. Prove that there exists exactly one value of x_1 for which $0 < x_n < x_{n+1} < 1$ for every n .

Sweden.

Time allowed: $4\frac{1}{2}$ hours

Each problem is worth 7 points.

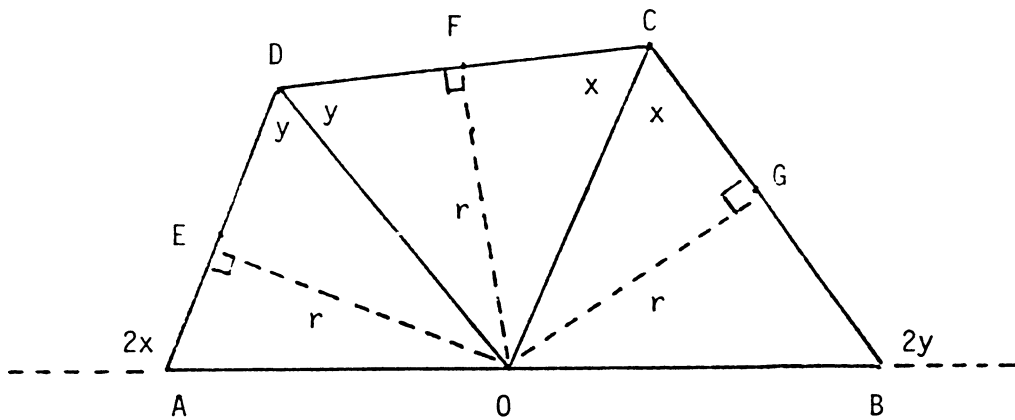
SOLUTIONS

These solutions are based on official solutions and will not be issued as they stand to schools who apply for IMO 1985 papers to the Mathematical Association.

1. Let the points of tangency be E, F and G as shown in the figure.

By elementary trigonometry, we obtain

$$\begin{aligned}
 AD + BC &= r(\cot y - \cot 2x) + r(\cot x - \cot 2y) \\
 &= r(\cot y - \cot 2y) + r(\cot x - \cot 2x) \\
 &= r \csc 2y + r \csc 2x \\
 &= AO + OB \\
 &= AB.
 \end{aligned}$$



2. The result is trivially true for $n = 2$. Suppose that the result is false and consider the smallest $n \geq 3$ for which there exists a non-constant two-colouring of $M = \{1, 2, \dots, n-1\}$ satisfying (i) and (ii) for some $0 < k < n$. In this example, let j be the least element of M which is coloured differently from element 1. There are three cases to consider.
- (a) $\underline{j > k}$. This is impossible since j and $j-k$ are required to have the same colour.
- (b) $\underline{j = k}$. The elements $n-k+1, \dots, n-1$ must have the same colour as element 1 but $k, 2k, \dots$ must have the opposite colour. This implies that $n-k$ is a multiple of k . But this is impossible since $k = j > 1$ and $(n, k) = 1$.

(c) $j < k$. Write $n = qk + r$, $0 < r < k$, and note that $(k, r) = 1$.

Consider the induced colouring of $M' = \{1, 2, \dots, k-1\}$ and note that it has the following properties:

(i) For each $i \in M'$, both i and $k-i$ have the same colour.

(ii) For each $i \in M'$, $i \neq r$, both i and $|i - r|$ have the same colour.

To see (ii), note that if $r > i$ then $i, n-i, n-k-i, \dots, r-i$ is a monochromatic sequence, and if $r < i$ then $i-r, k+r-i, \dots, n-i, i$ is a monochromatic sequence. Since the two-colouring of M' satisfies (i) and (ii) and is non-constant (1 and j have opposite colours), we have reached a contradiction to our choice of n as providing a minimal counterexample.

3. The proof is by induction on i_n . For $i_n = 0$ or 1 the result is trivial. Let R and S denote polynomials with integer coefficients. The following facts will be useful.

(a) If $k = 2^m$ then $(1 + x)^k \equiv 1 + x^k \pmod{2}$,

(b) If $\deg R < k$ then $w(R + x^k S) = w(R) + w(S)$,

(c) $|w(R) - w(S)| \leq w(R + S) \leq w(R) + w(S)$ (triangle inequality).

Let $i_n > 1$ be given and write $k = 2^m \leq i_n < 2^{m+1}$. There are two cases to consider.

(a) $i_1 < k$. Write $Q_{i_1} + \dots + Q_{i_n}$ as $R + (1 + x)^k S$ where

$R = Q_{i_1} + \dots + Q_{i_r}$ and $\deg R, \deg S < k$. Then

$$w(Q_{i_1} + \dots + Q_{i_n}) = w(R + S + x^k S) \quad (a)$$

$$= w(R + S) + w(S) \quad (b)$$

$$\geq w(R) \quad (c)$$

$$\geq w(Q_{i_1}) \quad (\text{induction hypothesis}).$$

(b) $i_1 \geq k$. Write $Q_{i_1} = (1+x)^k R$ and $Q_{i_1} + \dots + Q_{i_n} = (1+x)^k S$.

Then $\deg R, \deg S < k$ and we have

$$\begin{aligned}
 w(Q_{i_1} + \dots + Q_{i_n}) &= w(S + x^k S) && (a) \\
 &= 2 w(S) && (b) \\
 &\geq 2 w(R) && (\text{induction hypothesis}) \\
 &= w(R + x^k R) && (b) \\
 &= w(Q_{i_1}) && (a).
 \end{aligned}$$

4. Let $p_1 = 2, p_2 = 3, \dots, p_9 = 23$. For each element of M there is a corresponding vector (x_1, x_2, \dots, x_9) where $x_i = 0$ if, in the prime factorization of the element, p_i has an even exponent and 1 if the exponent is odd. By the pigeonhole principle, any subset of $2^9 + 1 = 513$ elements of M contains two distinct elements with the same exponent vector, i.e. two distinct elements whose product is a perfect square. It follows that M contains at least 737 pairs of distinct elements where the product of each pair is a perfect square.* Consider the square roots of these products. Again the prime factors are limited to p_1, \dots, p_9 and all we need are 513 to be sure that two have a product which is a perfect square. We thus find $a_i b_i = c_i^2$ and $a_j b_j = c_j^2$ where a_i, a_j, b_i, b_j are elements of M and $c_i c_j = d^2$, i.e. $a_i b_i a_j b_j = d^4$.

* To see this: Split the 1985 given integers arbitrarily into sets A, B, C containing initially 1472, 513 and 0 integers respectively. "Take a pair of integers with the same vector from B and place the pair in C. Transfer any 2 integers from A to B. Then B has 513 integers again and most contain 2 integers with the same vector."

Repeat the instructions in the sentences in quotation marks until A is empty. There will be then 737 pairs in C.

5. The common chords of the three pairs of circles are concurrent at their radical centre. Calling P the radical centre, quadrilateral $PMNC$ is cyclic (because $\angle PMN = \angle BKN = \angle NCA$). Thus

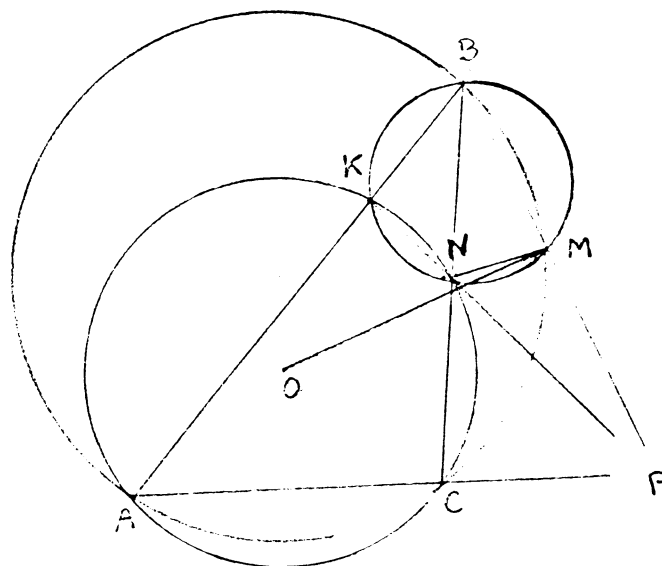
$$BN \cdot BP = BN \cdot BC = BO^2 - r^2$$

$$PM \cdot PB = PN \cdot PK = PO^2 - r^2,$$

where r is the circumradius of triangle ABC . Hence

$$\begin{aligned} PO^2 - BO^2 &= BP (PM - BM) \\ &= (PM + BM)(PM - BM) \\ &= PM^2 - BM^2. \end{aligned}$$

This proves that OM is perpendicular to BM .



6. Let $x_1 = x$. Then $x_n = P_n(x)$ where P_n is a polynomial of degree 2^{n-1} with positive coefficients. Note that P_n is increasing and convex for $x \geq 0$. Observing that the condition $x_{n+1} > x_n$ is equivalent to $x_n > 1 - \frac{1}{n}$, the problem can be reformulated as follows: there is a unique positive real number t such that $1 - \frac{1}{n} < P_n(t) < 1$ for all $n \geq 1$. Since P_n is increasing for $x \geq 0$ and $P_n(0) = 0$, it follows that for each n there are unique values for a_n and b_n such that $P_n(a_n) = 1 - \frac{1}{n}$ and $P_n(b_n) = 1$ respectively. Note that $P_{n+1}(a_n) = (1 - \frac{1}{n})(1 - \frac{1}{n} + \frac{1}{n}) = 1 - \frac{1}{n}$ and $P_{n+1}(a_{n+1}) = 1 - \frac{1}{n+1}$. Since P_{n+1} is increasing, it follows that $a_n < a_{n+1}$. Similarly, $P_{n+1}(b_n) = (1)(1 + \frac{1}{n})$ and $P_{n+1}(b_{n+1}) = 1$ so $b_{n+1} < b_n$. It follows that $[a_n, b_n]$ is a nested sequence of closed intervals and its intersection is non-empty. Since P_n is convex for $x \geq 0$ and $P_n(0) = 0$, we find that $P_n(x) \leq x/b_n$, $0 \leq x \leq 1$, in particular $1 - \frac{1}{n} = P(a_n) \leq a_n/b_n$. Together with the fact that $1 = b_1 > b_2 > b_3 > \dots$ this means that $b_n - a_n \leq 1/n$ for all n . Consequently, there is only one point t which belongs to all the intervals. This point satisfies $1 - \frac{1}{n} < P_n(t) < 1$ for all n and it does so uniquely. For any point $x \neq t$, either $x < a_n$ or $b_n < x$ for sufficiently large n meaning that $P_n(x) < 1 - \frac{1}{n}$ or else $P_n(x) > 1$.