Romanian Masters of Mathematics, 2009

Bucharest, 26 February – 2 March 2009

UK Leader's Report

The year 2008 saw the first appearance of a new event on the calendar of international mathematics competitions: the Romanian Masters of Mathematics. The UK team were winners of that inaugural contest, and we were extremely pleased to be invited to defend our title in 2009. For the second year running, the competition was held at the Tudor Vianu school in Bucharest. It was a tremendous success.

Each team competing in the Romanian Masters of Mathematics can have up to 6 members. On the basis of their performance in the British Mathematical Olympiad, the team selected to represent the UK was:

GBR1	Luke Betts (Hills Road Sixth Form College, Cambridge)
GBR2	Nathan Brown (King Edward VI Camp Hill School, Birmingham)
GBR3	Tim Hennock (Christ's Hospital, Horsham)
GBR4	Andrew Hyer (Westminster School, London)
GBR5	Peter Leach (Monkton Combe School, Bath)
GBR6	Craig Newbold (Whitley Bay High School)

The leader of the UK team was Adrian Sanders, formerly of Trinity College, Cambridge. The deputy leader was Sally Anne Huk, formerly of Haberdashers' Aske's School for Girls. The UK's participation was kindly sponsored by Winton Capital Management.

Ten teams competed this year: three from Romania (Rom A, Rom B and Vianu School team), along with teams from seven of the countries that have performed mostly strongly in the International Mathematical Olympiad (IMO) in recent years: Bulgaria, China, Italy, Russia, Serbia, UK and USA.

Unlike the IMO, this competition consists of a single exam. The paper lasts 5 hours and has 4 problems, all chosen to be roughly of the standard of the second or third problem of recent IMO papers.[1] Each competing nation was invited to submit two or three problems to be considered for the exam paper. For the second year running, one of the UK's problems was used. The other three problems came from Romania (2 problems) and Bulgaria.

[1] At the IMO, contestants sit two papers, each lasting 4.5 hours and each having 3 problems. These days, the problems on each of the two IMO papers tend to be in a steep ascending gradient of difficulty.

The problems of the Romanian Masters of Mathematics 2009 were as follows:

Problem 1. For positive integers $a_1, \dots, a_k$, let $n = \sum_{i=1}^{k} a_i$, and let $\binom{n}{a_1,\dots,a_k}$ be the multinomial coefficient $\frac{n!}{\prod_{i=1}^{k}(a_i!)}$.

Let $d = \gcd(a_1, \dots, a_k)$ denote the greatest common divisor of $a_1, \dots, a_k$. Prove that $\frac{d}{n}\binom{n}{a_1,\dots,a_k}$ is an integer.

(Proposer: Dan Schwarz, Romania)

Problem 2. A set S of points in space satisfies the property that all pairwise distances between points in S are distinct. Given that all the points in S have integer coordinates (x, y, z), where $1 \le x, y, z \le n$, show that the number of points in S is less than $\min\left((n+2)\sqrt{n/3}, n\sqrt{6}\right)$.

(Proposer: Dan Schwarz, Romania)

Problem 3. Given four points A_1, A_2, A_3, A_4 in the plane, no three collinear, such that

$$A_1A_2 \cdot A_3A_4 = A_1A_3 \cdot A_2A_4 = A_1A_4 \cdot A_2A_3,$$

denote by O_i the circumcentre of $\triangle A_jA_kA_l$, with $\{i, j, k, l\} = \{1, 2, 3, 4\}$.

Assuming $A_i \ne O_i$ for all indices i, prove that the four line A_iO_i are concurrent or parallel.

(Proposer: Nikolai Ivanov Beluhov, Bulgaria)

Problem 4. For a finite set X of positive integers, let

$$\Sigma(X) = \sum_{x \in X} \arctan \frac{1}{x}.$$

Given a finite set S of positive integers for which $\Sigma(S) < \frac{\pi}{2}$, show that there exists at least one finite set T of positive integers for which

$$S \subset T \text{ and } \Sigma(T) = \frac{\pi}{2}.$$

(Proposer: Kevin Buzzard, United Kingdom)

Each problem on the paper is marked out of a total of 7, so full marks for an individual would be 28 points. Romanian Masters of Mathematics is both an individual and a team competition. Medals are awarded to about half the competitors, roughly in the ratio 1 : 2 : 3 for gold : silver : bronze. Team performance is calculated by adding the scores of the three highest-scoring students from each country, so the maximum possible team score is 84.

The scores of the UK team were as follows:

	P1	P2	P3	P4	Total
Luke Betts (GBR1)	7	3	0	0	10
Nathan Brown (GBR2)	2	7	0	1	10
Tim Hennock (GBR3)	7	1	0	2	10
Andrew Hyer (GBR4)	5	3	0	0	8
Peter Leach (GBR5)	6	4	0	5	15
Craig Newbold (GBR6)	5	5	0	0	10

The medal boundaries for bronze, silver and gold were 11, 16 and 23 respectively, so UK won 1 bronze medal (and had no fewer than four other students who were one mark shy of getting a bronze medal). Six gold medals were awarded: three to students from China, and then one each to students from USA, Serbia and Bulgaria. China also won the team competition by an impressive margin. The final team positions and scores were:

1 China (74 out of 84)
2= Serbia, USA (60)
4 Russia (53)
5 Romania A (46)
6 Italy (45)
7= Bulgaria, Romania B (43)
9 Tudor Vianu School (36)
10 United Kingdom (35)

In purely *competitive* terms, the UK's performance in 2009 did not match the high water mark of 2008, but this is largely beside the point of such competitions. Through the warm hospitality and mathematical excellence of our Romanian hosts, the Romanian Masters of Mathematics 2009 was a first rate experience, mathematically and personally, for all involved. In order to give a flavour of the event for those who were not there, I will record a few memories of the competition.

Leader's Diary

To Bucharest

Our journey to the Romanian Masters of Mathematics 2009 will begin early on the morning of Thursday 26 February with a Wizz Air flight from London Luton, so we convene the night before at the Express by Holiday Inn adjacent to Luton airport. Craig and I are the last to arrive, by which time mathematical conversation has started to flow. There's Andrew, Peter and Craig whom I know from last year's Balkan Mathematical Olympiad; Tim who is the old hand of the squad as a veteran of the last two UK IMO teams; and

Luke and Nathan for whom this is the first time they are representing the UK in an international competition. There is also my redoubtable deputy Sally Anne, who has everything under control; she has brought with her the team polo shirts, which are a fetching powder blue.

The next day we start at the crack of dawn – breakfast at 05:45 and a short walk to the airport. The team is very grateful to Geoff[2] for having organised extra legroom for all. They occupy an entire row on the aircraft and keep their fellow passengers entertained with their mathematical analysis throughout the flight. We arrive on time in a sunny Bucharest – the only snow we will ever see is a few small piles of dirty leftovers from the blizzards of previous days. We are met by representatives of the host school and have our first experience of Romanian driving and traffic jams on the way to our hotel. This is the newly built and very comfortable Rin Grand on the south-eastern outskirts of the city. Apparently, it is the largest hotel in Europe, with some 1,459 guest rooms. At the IMO, contestants and team leaders stay strictly separated in different hotels until after the exams. One of the characteristically friendly features of this competition is that all the students and leaders will stay together in the same hotel throughout.

Between the journey and the time difference, we have missed lunch, so after checking in to the hotel, Sally Anne and I lead an expedition to the neighbouring *Carrefour* to buy provisions. In retrospect, telling a team of teenage boys to get as many snacks as they thought they might need for four days was an error. Sally Anne and I look with some doubt at the mountain of chocolate they bring hopefully to the checkout. In the event the catering at the Rin Grand was excellent.

By this time, the conversation has begun to include the games that were to entertain the team for the rest of the trip: Schnapps (Sally Anne and I still have not grasped how a few clicks of the fingers allow them to convey the identities of characters as diverse as Boadicea, Attila the Hun and Little Boy Blue); a guess-the-word game that included frequent chants of 'Contact... 3,2,1...'; and more.

The Problems

We head back to the hotel for dinner, where the students meet the other competitors for the first time. While they head off for a few games of 'Mafia' before an early night, I meet the other leaders, many of whom are old friends from olympiads past. I also receive the proposed exam that our Romanian hosts have compiled from the problems submitted by the participating countries. There is a shortlist of back-up problems in case the leaders spot any shortcomings in the draft paper. I am delighted to see that one of the UK's submissions has been pencilled in as Problem 4 for the contest. It is a really beautiful problem, in the spirit of the classical result

[2] Geoff Smith, the British Mathematical Olympiad Subtrust chairman

that any positive rational number can be expressed as an Egyptian fraction[3], but with a twist involving the arctangent function. The composer of this problem was Kevin Buzzard. The other problems on the draft paper look good to me too: a number theory problem involving multinomial coefficients at Problem 1; a beautiful plane geometry question at Problem 2; and a combinatorics problem at Problem 3.

As at the IMO, the leaders of the participating nations at the Romanian Masters of Mathematics constitute what is grandly called the 'Jury' for the competition. It is the Jury's job to agree the contest paper, the mark scheme and so on. At 08:30 the following morning (Friday 27 February), the Jury meets for the first time to finalise the paper. Kevin's problem goes through on the nod (Excellent!). So does the geometry problem. However, it is pushed back to position 3 on the paper, on grounds of difficulty[4]. After some discussion, the number theory problem proposed as Problem 1 is agreed. The main stumbling block is the combinatorics problem, which (after transposition with the geometry problem) is now slated as Problem 2. The leaders of Russia and USA volunteer that their students have seen similar problems before. So the proposed problem is set aside and replaced by one from the back-up list. The new problem is concerned with finite sets of lattice points in 3-dimensional space, and the Jury ponders several different versions of it before settling on a final one.

After working hard since breakfast, we are rewarded in late morning with a guided tour of the *Palatul Parlamentului* (Palace of the Parliament). The Ceauşescu regime began construction of this colossal building in 1983 with the intention that it should house all the major institutions of the Romanian state. Continuing the run of superlatives begun by Europe's largest hotel, we are told that the Palatul Parlamentului is the second largest administrative building in the world[5]. It certainly is vast, and it takes me 45 minutes to walk round the perimeter.

While I have been busy with the other leaders all morning, Sally Anne and the team have struck off independently to explore downtown Bucharest. They navigate the modern Metro successfully and have a pizza lunch in Piata Uniria, one of a number of large squares that grace central Bucharest. They report later that negotiating two taxis with a single 100 Lei note and not speaking a word of Romanian was challenging and, in hindsight, quite entertaining.

Before dinner it is the Opening Ceremony of the competition. Having won the inaugural Romanian Masters of Mathematics the previous year, the UK has donated an inscribed silver plate to go to the winning team

[3]An Egyptian fraction is one of the form $1/a_1 + 1/a_2 + \cdots + 1/a_k$ for some positive integers $a_1, a_2, \ldots, a_k$.

[4]The common convention on maths olympiad papers is that the problems are arranged in roughly increasing order of difficulty.

[5]... after the Pentagon.

each year hereafter. Sally Anne has brought it safely from the UK, and as part of the Opening Ceremony I am to present it to the Headmistress of Tudor Vianu school. We are sanguine about our chances of retaining the winners' plate, so before the ceremony we decide to get some photos of the team holding it while we still can. Photographing a silver plate is not as easy as one might hope. The plate is as shiny as a mirror, and I mostly succeed in taking pictures of my own reflection taking the photograph. The Opening Ceremony, at Tudor Vianu school, is very friendly indeed, filled with stories of the present and past of Romanian mathematics and their longstanding mathematical olympiad tradition. Romania were hosts of the very first IMO, 50 years ago, and many of the luminaries of Romanian mathematics are present to take a bow. The Deputy Minister of Education and a Romanian TV camera crew are also there.

The exam will be the following morning, so after dinner back at the Rin Grand, the team relax with a hot chocolate in the bar before bed. Romanian television is playing in the background, and we are pleased to see a lengthy report on the Opening Ceremony of the competition – the handiwork of the film crew we had seen earlier in the evening. The UK team feature repeatedly in the footage, looking dapper in their powder blue polo shirts.

The Exam

The team assemble in the hotel lobby at 07:30 (05:30 in the UK!) the next morning (Saturday 28 February) in order to be bussed the now familiar journey to the school. They are in good spirits, considering that they are about to sit a tricky 5-hour exam. As they wait outside the school to be called in to the exam hall, Peter apologises handsomely for the fact that he has lost his competition T-shirt and name badge. Someone points out that he is wearing both of them. Let's hope that this isn't an indication of Peter's level of brainpower this morning.

At maths competitions, the time when the students are sitting the exam is often the best opportunity for leaders and deputy leaders to do a bit of sightseeing on their own. Sally Anne and I seize this chance by going to visit two of the most beautiful old churches of Bucharest, the *Biserica Rusă* (Russian Church) and *Mănăstirea Stavropoleos* (Stavropoleos Monastery). Linguistically, Romania faces west, with a Romance language that is reminiscent, to our ears, of Italian. Religiously, however, the country looks east with an autocephalous Eastern Orthodox Church, to which denomination both of these splendid churches belong. Afterwards, Sally Anne and I walk a long route through the grand boulevards of the city centre. We pass by many of Bucharest's fine buildings, including the former Royal Palace, the beautiful *Ateneul Român* concert hall, and the lavish George Enescu Museum. Outside the Senate we see the *Memorial of Rebirth*, a monument to the fallen of the Romanian Revolution of 1989. It is striking to think that this occurred before any of our team were born.

After a fascinating morning we reconvene with the other leaders at the Buon Giorno restaurant for lunch. Then we head back to the Tudor Vianu school to meet the team as they come out of the competition. This is the moment of truth: will all of our team have managed to solve at least one question? Will any of them have managed the tricky geometry? Will any of them have managed the even trickier arctangents problem submitted by the UK? The first reports from team UK suggest that the answers to these questions are Yes, No and Yes respectively. Not too bad at all, but I fear our grip on the winners' silver plate may be slipping.

There is a short pause while scripts are photocopied. The copies go to the local coordinators[6]; the originals are given to me. While Sally Anne and the team head off after dinner for an excursion to 'Funland' amusement park, I remain at the hotel to work through the scripts and prepare for coordination the next day. First I look at the Problem 2 combinatorics, and it seems fairly promising. There's a complete solution from Nathan which is a lock for 7 out of 7. Craig's solution is almost perfect, but unfortunately he has changed n to $n-1$ at one point in the middle, which has slightly affected the rest of the proof. He will lose something for that, but it shouldn't be too much. The problem requires you to prove two separate upper bounds on a certain number. Luke and Andrew have proved the easier bound. Peter has only proved the harder one. So far, so good!

The scripts for Problem 1 number theory are a bit harder to understand. None of our team have spotted the neat two-line solution, but have elected to make their leader's life that bit more challenging by pursuing a variety of alternative methods, ranging from Luke's unimpeachable page-and-a-half to Andrew's perplexing five pages. What is he up to? I can't make it out. Knowing Andrew, there will be sense in it, but I can't quite trace the path through his various lemmas and simplifying constructions.

The geometry, sadly, is a bust. We are not going to get more than a mark or two there. Best news of all, though, is that we have a solution to the UK's own submission, Problem 4. Peter has cracked it, albeit with a few of the mathematical t's and i's not crossed and dotted. Of all our students, Peter has definitely had the best day. He has had most of the ideas to solve three of the four problems, which is a great achievement. Still scratching my head at Andrew's work on Problem 1, I crawl to bed at 02:30.

[6]At the Romanian Masters of Mathematics, as at IMO, marking of the exam is done by a process called *coordination*. Each team's leader(s) sits down with experts (coordinators) from the host nation to discuss their students' scripts and agree on a mark for them. The guiding approach is normally that marks are awarded for progress towards a complete solution. But what counts as non-trivial progress, and what proportion of a full solution a student has been able to complete are more often matters of judgement than of fact. This can make for interesting discussion. Coordination at this competition was a particular pleasure, because it was conducted in the same open and gentlemanly manner that our Romanian hosts brought to the whole event.

Results
Every year, 1 March is a special day in Romania. On this day, Romanian men mark the coming of Spring by handing every woman they know a mărţişor (a brooch with a red and white thread attached to it). Sally Anne is very touched to receive three of these brooches and also a small bunch of flowers from our friend Dan Schwarz, the chief coordinator. Sadly, there are not many girls competing in this contest – one from China, one from Serbia and a few from Romania – but they all get mărţişors too.

Over breakfast, Andrew explains to me how his solution to Problem 1 works. As I suspected, there was an underlying logic. But will I be able to sell it to the Romanian coordinators? They too were up late the previous evening, poring over the scripts. So when we convene at Tudor Vianu at 10:00, everyone is ready to get through the coordination in quick time.

First up for UK is Problem 2. Nathan pockets his 7 points. Craig loses 2 for mixing up n and $n-1$, which is fair. Tim gets 1 for proving one of the bounds in the analogous 2-dimensional problem. Now we have the scripts for Luke, Andrew and Peter, each of whom has done one half of the problem. At one stage, the intention was to split the marks 6 and 1 between those two halves; then 5 and 2; but after reviewing all the scripts, the coordinators have decided finally that it is going to be 4 and 3. This is good news for Luke and Andrew, who get 3 each for proving their half of the problem; but bad news for Peter, who gets only the 4 marks for proving the other half. I wonder whether he will be able to console himself with the fact the team gained in total from the change in mark scheme.

Next is Problem 4 where we agree on 5 marks for Peter's excellent work. We also pick up a welcome 2 marks for Tim for making a variety of pertinent observations without succeeding in pushing any of them through to a solution. Throughout the morning's coordination, I send through the UK's results to Sally Anne by text message. She and team are off on an excursion to the open-air Romanian Village Museum. Thankfully the weather is bright and sunny, so that they can appreciate the interesting displays of historical houses, farmsteads, stables, watermills and windmills from all of Romania's regions.

But there is no such recreation for me. I am straight on to coordination of Problem 1. This is going to be tricky. I just hope I can remember everything that Andrew explained to me over breakfast, which now seems like a long time ago. There are solid 7's for Luke and Tim; there is 6 for Peter, who has made a couple of minor errors. Craig's solution has a bit of a hole in it, which drops him down to 5; and we agree on 2 for Nathan. So we arrive finally at the *pièce de resistance* that is Andrew's script. The coordinator politely invites me to set out his approach. I begin by sketching in outline the four principal lemmas that underlie his solution, so that I can flesh out the bones a bit later on. The coordinator looks like a man who needs a bit more convincing, 'But he needs to prove all these!' – 'He does prove some of them, and the rest are obvious' – 'But why? And what does he mean here, here and here?', and so on. This goes on so long that I coordinate Problem 3 on the side for a bit of light relief (straight zeros, alas), before resuming with Problem 1. Eventually we converge on 5/7, which I think is absolutely right. I am very grateful for the painstaking attention that the Romanian coordinators have taken over this complicated but interesting script. Such things make all the difference to an olympiad competition.

By lunchtime, the results have been collated and the final rankings are in. Agonisingly, five of our six students are one mark shy of a medal boundary: Peter is one off a silver; Luke, Nathan, Tim and Craig are all one off a bronze. The team competition has been won impressively by China. USA and Serbia are joint second, with Russia in fourth. Congratulations to all of them – most of all, perhaps, to Serbia for managing to get up among three of the IMO giants.

All the medals are presented at a closing ceremony at Tudor Vianu school that evening; the silver plate goes to worthy winners China. After a farewell banquet back at the hotel, the team are off playing games again and socialising with their American and Italian counterparts till the early hours.

The next morning the coach for the airport arrives at 05:00 (a bracing 03:00 UK time) to collect us and several other bleary-eyed teams. The journey home passes without incident. We reach Heathrow 30 minutes early at 09:20, and the team split – weary but exhilarated.

Thanks...

Thanks to Luke, Nathan, Tim, Andrew, Peter and Craig for being such sterling representatives of the UK; to Rachel and all in the Leeds office for their help in preparations for the trip; and to Sally Anne for her constant good company and good sense.

We are very grateful for the support of Winton Capital Management, sponsors of the UK team for the Romanian Masters of Mathematics Competition.

Above all, thank you very much to our Romanian hosts, among whom I must mention Dan Schwarz ('Dan Schwarz is the problem selection committee') for his tremendous work leading the mathematical side of the competition, and Sever Moldoveanu and Radu Gologan for their endeavours on the logistics side. I think it must be by the generosity of spirit and intellectual accomplishment of such excellent people that the Romanian mathematical tradition flourishes so strongly to this day.