A Student's Thoughts on the Balkan Mathematical Olympiad

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Abstract

This report is intended to give the reader an insight into the atmosphere, goings on and mathematics at the 29th Balkan Mathematical Olympiad, which took place between the 26th April and the 2nd May 2012 in Antalya, Turkey. The competition was excellently run, and the trips were both thoroughly enjoyable – one was a boat trip on the mediterranean and the other involved tourist attractions such as an ancient ruins and a waterfall.

In the exam, the team did well – we achieved two silver medals, two bronze and two honourable mentions. Question 3 was our most well answered (collectively we scored 48), followed by Q1 (41), Q2 (33) and finally Q4 (16). The problems were all rewarding, and this report spends some time discussing them. However, the main focus of this report is on the various theorems and challenges the team worked on collectively in our spare time (of which there were many) – from theory of infinite sets to geometry and from inequalities to combinatorics, there seemed to be no end to the stream of problems supplied both by our leaders, Geoff and Gerry, and by ourselves.

Introduction

Like a fairytale, with its beginning, middle and end, most student reports are comprised of three elements, each with its own degree of interestingness. First, mathematical events; second, non-mathematical events; and third, stories about Adam Goucher. In his absence, I am left not with three elements, but with the relatively puny two. The reader is advised not to ignore the difference between three and two – indeed, if $3 = 2$ then $1 = 0$ (subtracting 2 from both sides) and as such I do not exist, which would be a crying shame for my similarly non-existent parents. Rather than bemoaning the lack of a third element, however, I shall embrace the opportunity to alter the structure of the student report entirely, by scrapping the 'non-mathematical event' section voluntarily and focussing entirely on mathematical events (effective from the next paragraph). Suffice to say that the hotel was luxurious, the weather impeccable, and our trip marred only by the lack of Geoff Smith (he was kidnapped by the organisers at the airport and taken to a secret location, wherein he was forced to work around the clock in unbearable conditions for 48 hours to prepare the paper, and only returned to us after the exam). From this point forth, minimal attention will be paid to all things non-mathematical – any parents reading this report should hence skip over sections densely populated with squiggles and remain in the realm of the ordinary. I have also appendicised solutions to some questions I set – some of these are in the answer section because you should actually try to find the solution first, and others simply so that readers with lesser mathematical ability (such as my mum) don't need to scroll through pages of equations to get to something they care about.

Although this report and that of Geoff Smith concern the same event, they are wildly different for three reasons. Firstly, he was in a different place about half of the time, either on the jury, marking or co-ordinating; secondly, his contains no maths whatsoever and mine contains very little else; and thirdly, while he handles the matter of our fellow passengers on the outward flight delicately and sensitively, I consider it my duty to make my disgust known.

Thursday 26th April

In which we journey to Antalya, discuss mathematics, are separated from Geoff, meet our guide and are given various freebies

Gatwick Airport on a working Thursday is, consistent with expectations, quiet. This result is easily derived from some basic set theory, starting with the set S of people in the UK. Allowing for a handful of miscellaneous exceptions, such as ourselves, we define the following:

- the set of working people, W
- the set of noisy people, N
- the set of retired people, R

- the set of people at Gatwick on a working Thursday, G

Now $N \bigcap R = R \bigcap W = W \bigcap G = \emptyset$. So $\forall\, g \in G$, $g \notin W$, so $g \in S \setminus W$. After the harmless but perhaps derogatory assumption that $W \bigcup R = S$, so $S \setminus W \bigcap S \setminus R = \emptyset$, we get $g \notin S \setminus R$, so $g \in R$. Therefore $g \notin N$ and we have explained the quietness of Gatwick. One might predict that our flight would be quiet by the same logic. One would be very wrong. The flaw in the logic is the premise that $W \bigcap G = \emptyset$. In fact, $W \bigcap G = C$ – the non-empty set of people who, while they do work, have neither interest in or commitment to their occupations, nor desire to contribute to society, in sufficient quantities to pay slightly more for their vacations by waiting for school holidays – indeed, C is the set of chavs.

Despite the adverse conditions (my seat, for example, is an armrest for two fat men, both of whom smell of beer and urine) we manage to get some maths going. Gerry (deputy team leader) grapples with some beautifully drawn geometry while Geoff (team leader) tells us stories of past competitions. Matthew (UNK4) works on a selection of old BalkanMO problems and later some infinite set theory. Matei (UNK5) and Daniel (UNK3), with occasional help from Robin (UNK1) and I (UNK2), work on an interesting set of geometry problems involving cevians.

It is well known that for any point P coplanar with triangle ABC, where AP meets BC at D, BP meets AC at E and CP meets AB at F,

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$$

The question concerns situations where $AF - FB + BD - DC + CE - EA = 0$. Of course, the trivial examples were the centroid, the gergonne point and the nagel point, but interesting points to consider were the incentre, orthocentre, symmedian point or circumcentre. Given that P is each of these, determining the properties of triangles which comply with the above condition is a worthwhile exercise. It generally involves finding expressions for each relevant line segment (I use areal co-ordinates), then forming a huge polynomial (generally in a, b and c) and factorising. Importantly, all the terms in the polynomial had the same "order" (here the word means the sum of the powers of each of a, b and c – for example, a^5bc is 7, as is a^4c^3), and Matei teaches us a nice way of factorising such polynomials. If the "order" of each term is n, draw a triangle of isogonal points with $n+1$ points on each side. For example, for $n = 3$, 10 dots should be arranged in the form of a ten-pin bowling setup. Label each vertex a, b and c; each point refers to one term. The distance from a dot D to each edge (as a stepping-stone distance, not as the crow flies) is the power of the letter assigned to the opposite vertex in the term referred to by D. Over each dot, write the coefficient of that term (including the sign). Now it is possible to pick out factors by noticing repeating patterns – for example, $a^3 - a^2b + ab^2 - b^2c + bc^2 - c^3$ would be written as

$$\begin{array}{ccccccccc}
 & & & & a & & & & \\
 & & & & 1 & & & & \\
 & & & -1 & & 0 & & & \\
 & & 1 & & 0 & & 0 & & \\
 & 0 & & -1 & & 1 & & -1 & \\
b & & & & & & & & c
\end{array}$$

and it is relatively easy to spot the factor

$$\begin{array}{ccccc}
 & & a & & \\
 & & 1 & & \\
 & 0 & & -1 & \\
b & & & & c
\end{array}$$

and its negative version, which gives the factor $a - c$. To someone good at factorisation, this may seem like a waste of time, but as kings and queens have said, I am not good at factorisation.

Sitting in the middle on the six, I have the privilege of floating from the geometrical musings of Daniel and Matei to Matthew's set theory. He is considering pairs of uncountable sets of reals between which a bijection is possible that preserves order. We agree that such bijections exist regardless of size, but only provided the intervals' ends are closed and open in the same way (intervals starting or ending with $-\infty$ or ∞ are taken as open on that end). We turn our attention to the cantor set, agreeing – falsely – that the map is possible; write each number in the cantor set base three, and only 0s and 2s are used (1 is written as $0.222\ldots$ and 0.1 as $0.0222\ldots$ and so on). Then turn each two into a 1 and we have 0 to 1 in binary. Unfortunately, as I realised at lunch on Friday (as I write) a third and two thirds both map to a half, as $0.0111\ldots = 0.1$ in base two. We now turn our attention to sets that we will suspect we will get nowhere with – for example, a bijection between $\mathbb{R}$ and $\mathbb{R} \setminus \mathbb{Q}$ preserving order – I cannot find a way through this at all. We ask Geoff, and he confirms that it's "hard" (he later proves that no such bijection exists, as does Robin[1]) and steers the conversation towards maths that's way over our heads – a problem about aeroplanes. We are to estimate the distance to the horizon in a plane at cruising altitude, and I calculate that it's somewhere between 200 and 300 miles. Geoff nods, and concludes that when flying over London, Manchester is visible. Matei and Daniel have turned their attention to the Butterfly Theorem, a result which they would focus on throughout the following day. On the plane it interested me as well, but at the time of writing I am sick to the back teeth with this useless, rancid, boring and pathetic result, so you can look it up.

Our rowdy flight-mates have added *angry* to the list of unpalatable traits they exhibit, as the plane has run out of alcohol in the first half of the flight (I wonder if they realise that this was their doing entirely?) but are sufficiently intoxicated to not really care. A pair of totally hammered girls strike up a conversation, and are happy to discuss the impressive achievements of Pythagoras when they find out who we are and where we're going. I breathe a sigh of relief when the plane lands, and say a short prayer that in their drunken stupor they will forget my name and therefore be unable to follow up on their oath to add me on Facebook. We say a teary farewell to Geoff, and meet Salih, a friendly Turkish student with impressive English.

Gerry constructs the reals on the journey to the hotel, starting with Peano's successors and culminating in the Dedekind cut which launches us from rational to real (the only part of the process we hadn't foreseen). Soon we pull up at the hotel.

At this point the reader should become acquainted with the notes document that the team was sent a few days before we left – based, according to Geoff, on observations from Moldova two years before – such tips as 'bring towels', 'bring loo roll', 'do not drink tap water' and 'do not leave your valuables lying around obviously in hotel rooms whilst you are out'.

We are staying in a five star hotel. There are six main dining areas, each heaving with mouthwatering turkish specialities. There are three swimming pools, the main one comprised of two large sections joined by a decorative channel under a bridge, and including four water slides, a statue of a mermaid and a fountain. Hoteliers are on hand to carry our luggage to our rooms, each of which could house another swimming pool, and includes a large en suite bathroom and a balcony (I beat Robin to the double bed; selfish since I am noticeably smaller).

If you choose, at this point, to berate me for fickly forsaking my promise to avoid non-mathematical events, then you lack foresight. It's time for a chat about Hilbert's Hotel. A hotel with $\aleph_0$ rooms is simply not impressive to us any more, having seen our enormous hotel. Admittedly it does not have $\aleph_0$ rooms, but we agree that it is only a few dozen short. The Continuum Crib, on the other hand, is a hotel (designed by ourselves) with $2^{\aleph_0}$ rooms (each assigned a real number between 0 and 1). Cantor regularly receives continuum coaches, and gives each passenger a room (guests never leave – the continuum crib should really be called Gabriel Gendler's great big graveyard). Of course, this is no problem; Cantor can house $2^{\aleph_0}$ of these coach cohorts by a rich choice of $\mathbb{R}^2 \to \mathbb{R}$ bijections, so $\aleph_0$ of them (they do not arrive continuously and so are countable) shouldn't be a problem. Cantor has always operated as follows – each time a coach arrives, he assigns it an integer ID (starting at 1 and counting up). He then assigns each passenger a new real number between $1 - \frac{1}{2^{n-1}}$ and $1 - \frac{1}{2^n}$, and sends each guest to these rooms. Unfortunately, Cantor's ghoulish guests are getting upset. Although Cantor knows that each guest has its own real number, and that the reals are evenly spread, the guests seem to think that as the interval shrinks, they become more crammed. Can Cantor change his accommodation system to avoid this perception, without making his guests change rooms after they have arrived?[2]

We unpack our suitcases and rummage through our BalkMO freebies, including turkish delight, which is turkish, and a pair of compasses, which are not (also included is a list of these freebies which does not include itself, but Cantor *and* Russell seems too much for one evening). We get to bed at 2 in the morning, asserting that considering the luxury that is our hotel, Geoff must be in Paradise.

Friday 27th April

In which we explore the hotel, attend the opening ceremony, discuss mathematics and swim

Despite today's date being the $3^{3^{\text{th}}}$ of the $2^{2^{\text{th}}}$, very few mathematical events occur. We wake at 7.45 and view the hotel in daylight for the first time – indeed, it is a thing of beauty, and it is vast.

Robin and I have the best view from our window, overlooking the pool(s) and the mediterranean. Bizarrely, a river runs along the coast without meeting it for quite some distance, but nobody really cares. An early transfer moves us to the opening ceremony, which takes place in a third hotel, even more luxurious than ours – the whole thing is an enormous golfing resort, much like most of Antalya. I will not detail the events at the opening ceremony, since they all come under the category of non-mathematical event.

We now have 10 hours of free time – a wonderfully relaxed, calm day. Matei, Robin and I swim for a while, although this too is not very mathematical, but for most of the afternoon all of us (except Harry) sit by the pool doing maths. Daniel and Matei independently solve the Butterfly Theorem, and Gerry shows us some beautiful problems involving symmedians. The first is a known result of symmedians, but Gerry wants a geometric proof. A triangle ABC is circumscribed by Γ. Tangents to Γ at B and C meet at T. prove that $\angle TAB = \angle CAM$, where M is the midpoint of BC.[3]

We all marvel at the beautifully symmetric proof he gives – I am particularly impressed with the construction of two new points, since I am rarely able to use constructions in my proofs. Now Gerry shows us a wonderful symmedian problem from the Australian Olympiad: in a triangle ABC, Γ is the circumcircle, M is the midpoint of BC, H is the orthocentre and D is the foot of AH. MH meets Γ above BC at X; XD meets Γ again at Y.

Show that AY is a symmedian!

I spend most of the day writing the student report so far, and updating the others with my progress every so often. As I write (today) the Australian problem remains unsolved.

Saturday 28th April

In which we sit the exam, are reunited with Geoff, discuss our solutions, play chess and discuss mathematics

With our joyous rest day a distant memory, we wake up bright and early for the exam. This is to begin at 9 and to last 4.5 hours, much like FSTs, NSTs, XSTs, IMOs, RMMs and plane journeys to Turkey. There will be four questions (1 more than in the exams listed above, and as we know, the difference between 4 and 3 should not be ignored) and as such the exam is a daunting undertaking. Geoff mentions that last year was a tough one – set in Romania, a country of talented mathematicians and perhaps more talented problem setters – and hopes that this year would be easier. He is correct – the problems (as follows) result in unusually high marks across most countries – the Turkish team, for example, score 40, 40, 40, 39, 36 and 31. The problems are as follows:

Question 1 – Let A, B and C be points lying on a circle Γ with centre O. Assume that $\angle ABC > 90°$. Let D be the point of intersection of the line AB with the line perpendicular to AC at C. Let ℓ be the line through D which is perpendicular to AO. Let E be the point of intersection of ℓ with the line AC, and let F be the point of intersection of Γ with ℓ that lies between D and E. Prove that the circumcircles of $\triangle BFE$ and $\triangle CFD$ are tangent at F.

Having the proficiency in Geometry of a coconut, I manage to miss this problem completely, failing to notice even some of the most blatantly obvious properties of the configuration – however, my able colleagues are less unsuccessful. Matthew runs out of time half way through his proof, but the other UNKs claim complete solutions. Harry's solution (without doubt his best) is detailed in the answer section.[4]

Question 2 – show that for all $x, y, z \in \mathbb{R}^+$,

$$\sum_{\text{cyc}} (x+y)\sqrt{x+y}\sqrt{y+z} \geq 4(xy + xz + yz)$$

Robin, Matei and I have the most complete solutions here, all using the substitution $a^2 = x + y$, $b^2 = x + z$ and $c^2 = y + z$. Robin then employs a useful inequality from Schur, followed by Muirhead. His solution is akin to using a nuclear bomb to apprehend a shoplifter – it is ridiculously powerful, and he obliterates the inequality like a psychotic pyromaniac. In other words, it is incredibly professional.[5]

Matei's solution is extremely insightful – after the initial a, b, c substitution, he carries out yet another, with $d + e = a$, $d + f = b$ and $e + f = c$. After an enormous expansion (which he doesn't carry out line by line, to the dismay of Gerry who must therefore slog it out himself when marking the paper) this becomes a sum of squares and Matei is done. I am enchanted by this solution. Like a demented wombat in comparison to Matei and Robin, the elegant kangaroos, I opt for a bizarre and unusual proof, which is imaginative but drastically more complicated. Nevertheless, it makes for some wonderful comedy, so I have included it in the answer section.[6]

On the subject of solutions to question 2, the answer section below contains two more – one from AoPS[7] that afternoon and the other from Daniel the following day[8] (he is disappointed as Q2 is the only one he doesn't get, and then after the exam he finds the best solution of them all).

Question 3 – let n be a positive integer. Let $P_n = \{2^n, 2^{n-1} \cdot 3, \ldots, 2^{n-r} \cdot 3^r, \ldots, 3^n\}$. For each subset X of P_n, we write S_X for the sum of elements of X, with the convention that $S_\emptyset = 0$. Suppose that $y \in \mathbb{R}$ with $0 \leq y - S_Y \leq 3^{n+1} - 2^{n+1}$. Prove that there is a subset Y of P_n such that $0 \leq y - S_Y < 2^n$.

This "combinatorics" problem (it was the only one on the shortlist, so we suspect that the organisers were desperate) proves to be our strongest question, and we all submit roughly the same solution. Matthew's is recorded in the answer section.[9]

Question 4 – Let $\mathbb{Z}^+$ be the set of positive integers. Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that the following conditions hold:

i) $f(n!) = f(n)!$ for all $n \in \mathbb{Z}^+$

ii) $m - n \mid f(m) - f(n) \; \forall \; m \neq n \in \mathbb{Z}^+$

Daniel claims the only solution here (although Matei gets a long way). The answer, of course, is $f(n) = 1$, $f(n) = 2$ or $f(n) = n$ – an observation which would have earned me a mark had I bothered to write it down – I suppose this kind of mistake is what the Balkans are for. I cannot envisage myself coming up with Daniel's proof in a million years – the number of steps is ridiculous.[10]

Upsettingly for anyone who cares about the environment, we are only allowed to write on one side of the paper (they may as well have given us möbius strips). Another interesting observation is that the rules sheet includes a prohibition against having toys in the exam hall – I wonder how much help even the most intelligent of teddy bears could realistically be? We troop out of the hall (which lost power on two occasions during the exam) and are reunited with Geoff, who debriefs us. We now have a free afternoon (followed by three free days) in which to do some maths. After a spot of giant chess, in which I pose as a knight, we consider some colouring problems. Daniel recalls FST1 Q2 – which I must be vague about since the paper is not yet on line, so this will only make sense for those who sat the paper – and wonders whether an optimal solution is possible with a diagonal pair; the upper bound required in a solution theoretically allows it. Two 3-colourings contradict each other nicely, giving the desired result.

Over dinner, Daniel shows us a beautiful problem wherein inversion is actually intrinsically useful, rather than simply a reframing of the problem. This is from Josh Lam on AoPS: AB is the diameter of a circle Γ. P is a point on AB and Q is a point on Γ such that $\angle QPA = 90°$. The largest possible circle is drawn that fits inside the bisected segment QPB. This is tangent to AB at G. Show that $AG = AQ$. The reader is encouraged to think before checking the answer.[11]

Sunday 29th April

In which we discuss mathematics, travel to Perga, see a waterfall and visit the city of Antalya

Today we journey to Perga, or as Geoff refers to it, "Apollonius's crib". Quite how and when he learned jive remains a mystery. The outing will, of course, disrupt our mathematical investigations, so we get some maths going very early. In bed before breakfast, I consider the geometry problem from the plane – points P for triangle ABC with cevians APD, BPE and CPF such that $AF - FB + BD - DC + CE - EA = 0$. I attempt to find an equation for the locus of points that meet the condition given a triangle with sides a, b and c – P with areal co-ordinates (x, y, z) is effective if and only if

$$a(z-y)(y+x)(x+z)+b(x-z)(z+y)(y+x)+c(y-x)(x+z)(z+y) = 0$$

which I will mess around with later. Harry observes that the gergonne point is the symmedian point of the intouch triangle by Gerry's symmedian proof from Friday. Over breakfast Daniel explains his fifth solution to question 2, which I mentioned in yesterday's report (answer 8) – it uses a remarkable yet well known factorisation, and I wonder why none of us spotted this in the exam.

We do an awful lot of trigbashing, showing that $\tan A + \tan B + \tan C = \tan A \tan B \tan C$ where $A + B + C = 180$, that $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$ and (for posterity, nobody found this useful or non-trivial) that $\frac{\sqrt[3]{a^2 b^2 c^2 \sin A \sin B \sin C}}{2}$ is the area of a triangle.

Geoff gives a cute geometric representation of my identity that $\sin A + \sin B + \sin C \geq \sin 2A + \sin 2B + \sin 2C$ for acute triangles (read my solution to Q2 from yesterday to understand how this fits in). In an acute triangle, we consider the orthic triangle DEF. Simple trig gives the sides of DEF as $R \sin 2A$, $R \sin 2B$ and $R \sin 2C$, whereas the larger triangle has sides $2R \sin A$, $2R \sin B$ and $2R \sin C$ – we show that $BC \geq FD + DE$ cyclically (by BMO1 Q6 2011) and we're done. Attention switches to whether this is also the case for obtuse triangles – my trigonometric proof involves the fact that $\sin x$ increases and $\cos x$ decreases as x increases between $0°$ and $90°$, whereas Geoff's orthic triangle proof doesn't hold for more complex reasons which I won't detail. Instead, Geoff gives a lovely proof – starting with an acute triangle (not equilateral) the inequality holds strictly and this triangle can be transformed continuously until it is any other triangle – the difference between the LHS and the RHS changes continuously, never crosses 0 (this would require it to become equilateral) and starts positive, so the inequality must hold no matter what the triangle. I love this proof as it is incredibly classy and simple, although slightly more work is required to show that an equilateral triangle is indeed the only one for which equality holds. A safer proof (also from Geoff) is recorded in the answer section.[12]

Some thought is given to the pedal triangle and the wonderful formula for its area based solely on the power of P to the circumcircle. Given P coplanar with triangle ABC, the area of DEF, where D, E and F are the feet of the perpendiculars from P, is given by

$$[DEF] = \frac{R^2 - OP^2}{4R^2}[ABC]$$

I remark that a result of this is the well known fact that the feet of perpendiculars from a point on the circumcircle are colinear. Geoff outlines a proof that goes over my head.

The trip itself is fairly mediocre, so I won't bore you with the details, but let it be known that there was a greek ruins, a waterfall, a restaurant, a city and a bus.

There's surprisingly little time to relax when we return to the hotel, but enough to play a few games of pool (we observe, in the spirit of a maths competition, that not all points on the table are coplanar and that not all points on the cue are colinear, which ruins things somewhat). We also find out that Salih's name is not Salih. I make the utterly useless observation that a triangle and it's medial triangle's orthic triangle's intouch triangle share a circumcentre.

As I come to this realisation, I begin to experience a stomach ache. I feel increasingly unwell over the next hour, and the pain builds to complete agony. Sensing imminent danger, I uncurl myself from a ball-like position on my bed and assume, as a safety precaution, a more appropriate position in the necessarium. I re-emerge 10 minutes later having dealt a serious blow to the toilet's self esteem. My stomach ache is completely relieved and I fall asleep very quickly; the only health scare of the trip for any of the UNKs has passed.

Monday 30th April

In which I almost kill Robin, we play poker and pool, swim, discuss mathematics and find out our marks and medals

Today is completely free. I am awoken at a luxurious 9 o'clock by the usual pillow to the face. In my half-awoken haze my pillow-throwing response is less refined than usual and I come close to unhooking a large painting from the wall, which would have made my least professional pillow-throw additionally my most deadly, since Robin is directly below. We breakfast at the similarly luxurious 9.15, allowing for a not so luxurious 15 minutes to get dressed, wash and shower. Over breakfast we receive our marks for question 1 – Matei, Harry and Daniel score 10; Robin makes 8 after being docked 2 for failing to show that E is the orthocentre. Matthew takes a respectable 4 after not finishing his proof, and I am surprised to pick up a mark just for recognising the relevance of the point opposite A over O. This seems extremely generous considering Geoff's warning that I may be fined a mark on Q2 for not showing that the triangle I have constructed is real by triangle inequalities, despite showing that it is acute. How there could be an acute triangle that is not a triangle I do not know, but co-ordinators will be co-ordinators.

After an extensive game of poker, which is admittedly quite boring, we relax for an hour and then regroup for lunch. Geoff meets us and tells us our scores for Q2 – 10 for Robin and Matei, the Cranch Expected Score of 0 for Daniel and Matthew, and 4 for Harry for his substantial progress. My score is still undecided, and Geoff explains that while it was accepted that very few acute triangles are not triangles, my final step using the rearrangement inequality was not symmetrical. The problem was that instead of using $\sin A\cos B+\sin B\cos A\geq\sin A\cos A+\sin B\cos B$ as I wrote in this report, I had foolishly written $\sin A\cos C+\sin B\cos B+\sin C\cos A\geq\sin A\cos A+\sin B\cos B+\sin C\cos C$ and the introduction of the middle variable means that it doesn't work cyclically. Luckily, as I find out during a farcical game of pool, my fine is just 1 mark since rearrangement would work with such a minor adjustment. As such I have gained an undeserved mark and carelessly lost one, leaving me on track for my predicted 20 marks – the Lord giveth and the Lord taketh away (Job 1:21). We await our marks for Q3, which should be strong – interestingly, we notice that while we all found Q3 easier than Q2, suggesting that the jury were mistaken to place them in that order, the marks of other countries prove us wrong. Perhaps the British squad needs more work on inequalities?

While the swimming pool itself offers little in the way of maths besides aerodynamics and the mechanics of water slides, the deck chairs are a hub of discussion over the pinnacle of mathematical challenges – AQA C2. Such tests of the mind as "given $\sin\theta+\cos\theta=0$, show that $\tan\theta=-1$". Whoever defined this as mathematics was wrong. In view of this fact we consider establishing an A-level in Olympiad mathematics, but eventually decide that it would be far too hard for the ordinary mathematician. Our scores for question 3 arrive – Robin scores 8, Matei 9, and the rest 10 other than Harry, who missed Q3. Q4 gives Daniel 10, Matei 4 and Robin and Matthew 1. We're all fairly happy with our scores, but predict that each medal boundary will be precisely 1 above an UNK score. Geoff tells me that my Q3 proved a pain in the backside again, due to a language issue, but clearly this has been resolved without a fine. Attention then turns to whether any interesting OEIS sequences contain our scores – then we attempt to find a quintic polynomial p such that $p(x)$ is the score of UNK x for all integer x from 1 to 6. We can't be bothered, and it seems that Wolfram Alpha can't be bothered either. At this point the patient reader will find out why the report is so long – it is a trial of patience for our benevolent dictator, James Cranch. If, in fact, he has read this far, then we congratulate him and apologise for mentioning that Guernsey FC currently reside in the 9th level of the English Football League.

We head to the beach, discussing an Advanced Mentoring Scheme problem involving an infinite number of points in general position coloured green and black (with at least 1cm between each to avoid a continuum of points) – we are required to show that in no such colouring can every triangle of green points contain a black point and vice versa. My proof is a bash, but Matei's is gorgeous.[13]

Seeing that kayaking was closed from 5 o'clock (the time now being 5 past 5) I head back to the poolside, where Gerry is talking about dodgy teaching of maths at A-level, such as poor definitions of integration and differentiation, and sloppiness with inequality manipulation. For example, he solves:

$$\frac{1}{2}^n < \frac{1}{1000000}$$

$$\log_{\frac{1}{4}} \frac{1}{2}^n < \log_{\frac{1}{4}} \frac{1}{1000000}$$

$$n < \frac{\log_{\frac{1}{4}} \frac{1}{1000000}}{\log_{\frac{1}{4}} \frac{1}{2}}$$

which, of course, is wrong (the sign is the wrong way round) – but which step went awry? $\log_{\frac{1}{4}} \frac{1}{2} > 0$ so no multiplication by negatives has occurred.[14]

He then integrates $f(x) = 1/x$ between -2 and 5 (tut tut) by $\lim_{\varepsilon \to 0}$ of the integral between -2 and $-\varepsilon$ plus that between ε and 5 (defining $\ln -x$ correctly). Using 4ε, or $r\varepsilon \; \forall \; r \in \mathbb{R}$ in the positive version gives a different result, which explains part of the definition of differentiability – that

$$\lim_{\varepsilon \to 0} \frac{f(x+\varepsilon) - f(x-\varepsilon)}{2\varepsilon} = \lim_{\varepsilon \to 0} \frac{f(x+r\varepsilon) - f(x-\varepsilon)}{(1+r)\varepsilon} \; \forall \; r \in \mathbb{R}$$

or something along those lines.

Finally he completes the sequence 3, 0, 0, 4, 2, 0 with 12. Genius.

Robin and I have another hack at least year's inequality – for $x+y+z = 0$:

$$\sum_{\text{cyc}} \frac{x(x+2)}{2x^2+1} \geq 0$$

To my shame, I multiply by the denominators and expand out, but this doesn't help. I try substitutions which also fail. Although I do not solve the problem on the trip, I have more success a few days after getting home – see the half-cooked answer below.[15] We spend the evening playing lots of pool, even more table tennis and a ridiculous amount of maths. At the centre of attention, among several other integer series with little founding in traditional mathematics, is this ghastly thing:

$$1, 1, 2, 1, 1, 2, 2, 2, 3, 1, 1, 2, 1, 1, 2, 2, 2, 3, 2, 1, 1, \ldots$$

in which each number is found by counting the number of full repetitions of the longest repeating sequence finishing on the previous number, starting with 1. With $f(n)$ defined as the term of the above sequence at which n

first appears (so $f(1) = 1$, $f(2) = 3$, $f(3) = 9$ and so on) it is agreed that $f(n)$ grows rapidly. Very rapidly. Matei conjectures that 10 has not been reached by term G, where G is Graham's number (whose best approximation is the number of rooms in our hotel, although it may be slightly greater if we're being honest). I agree that $f(10)$ is large, but not that large. This dispute is quickly settled when Harry corrects Matei's definition of Graham's number, which was too small by a factor of roughly Graham's number (for anyone who cares, Matei is thinking about $g(1)$, and Graham's number is $g(64)$, so Matei is out by at least 63). Geoff and Gerry finally appear with the medal boundaries – 30 is Silver, 20 is Bronze and Gold is 39 due to the relatively easy paper. This means that Daniel and Matei take Silver, and Robin and I Bronze. Contrary to our expectations, two of us land luckily on a medal boundary, whereas we suspected that multiple UNKs would fall a point shy. I note that giving each team member a number between 1 and 6 according to relative rank, all three bedrooms add to 7. Further, in reverse order of UNK, each value oscillates towards 3.5 – 6, 1, 5, 2, 4, 3. These two are closely related, as rooms were assigned alphabetically, and so were UNK numbers. Just before bed, Robin explains a fantastic trig identity – for a triangle ABC, there is a point P (the Brocard point) such that $\angle PAB = \angle PBC = \angle PCA = \omega$, and for this point $\cot A + \cot B + \cot C = \cot \omega$, which is impressive.

Tuesday 1st May

In which we discuss mathematics, relax on a boat, attend the closing ceremony, escape from the closing gala and write Salih a letter

Today is our last full day, and our second road trip – or rather, sea trip – we will spend the day on a boat. As we congregate outside reception to board coaches to the harbour, Daniel, Robin and I tackle the cotangent/brocard point problem (as well as its corollary that $\omega \leq 30°$).[16]

We consider the inequality from Balkans 2010:

$$\sum_{\text{cyc}} \frac{a^2 b(b-c)}{a+b} \geq 0$$

I solve this the predictable, reliable and fast way.[17] Daniel, as usual, puts my method to shame[18] (I advise that you take a look). Despite the beauty of Daniel's solution, I do not regret doing what I did. It is efficient, it is safe and it is easy. I see no reason to bother with beautiful solutions to ugly inequalities. They are horrible, and deserve to be hit around the head with a machete, and that's what I endeavour to do. In the spirit of this entire event, geometry quickly becomes the main subject of the day (at this point we are on a boat cruising through the extraordinarily blue mediterranean). Gerry sets a problem involving an orthogonal tetrahedron with $\angle ABC = \angle CBD = \angle DBA = 90°$ – show that $[ABC]^2+[CBD]^2+[DBA]^2 = [ACD]^2$. This yields to a quick bash. Geoff then sets two problems – first, of a regular tetrahedron $ABCD$ with any point P on its circumsphere – show that $AP^4+BP^4+CP^4+DP^4$ is fixed. The second involves a triangle with incircle Γ. The tangent to Γ parallel to BC meets AB at A_1 and AC at A_2; B_1, B_2, C_1 and C_2 are defined similarly. It is required to prove that

$$\sum_{\text{cyc}} AA_1 \cdot AA_2 \geq \frac{a^2+b^2+c^2}{9}$$

These two prove more difficult, but we are soon joined in our quest by a band of Romanians and Turks with an average BalkMO score of 40. I am proud to show them something they don't know – Matei's factorising triangle. The second is quickly dealt with but the first is still on the table as the team is disbanded by an enormous waterfall.

Over lunch, Geoff reveals his motives – he has been demonstrating the useful nature of the Huygens-Steiner theorem, which he asserts can solve anything which involves even powers of lengths. He then explains what Huygens-Steiner is (this is the second time I've heard an explanation of Huygens-Steiner and I am beginning to feel as though after five more explanations I will understand it). Although the first solves itself in co-ordinates and the second can be tackled with plane euclidean geometry, I will omit these proofs because the point of the exercise was to use Huygens-Steiner, and I cannot write up a Huygens-Steiner proof because I don't understand Huygens-Steiner. The conversation flips to useless geometry, and Geoff mentions Adam Goucher's favourite – the product of the four tritangential radii is the square of the area. In other words,

$$[ABC]^2 = r \cdot r_A \cdot r_B \cdot r_C$$

This is proved with Heron. I also realise that ABC is the orthic triangle of $I_A I_B I_C$ – a pleasing but even more useless result. Gerry shows us a nice trick – writing Pascal's triangle mod 2 and reading each row as an integer in binary, we get the sequence 1, 3, 5, 15, 17, 51, 85... which is the sequence of constructible n-gons up to multiplication by a power of 2.

Unfortunately the boat ride must end and we are forced from the cool sea onto stifling buses with little air-con. Luckily, Gerry is always a source of fascinating maths, and he teaches me about ordinals, ω, ε_0 and transfinite induction. Some of it goes over my head, but what I understand, I really enjoy. I won't go into detail here, but I recommend that the reader ask Gerry for some very nice observations. Orders of ω^2, ω^3 and ω^n are not too hard to find, but ω^ω proves more difficult. Eventually, Matei identifies one: in the continuing sequence

$$a_1 + \cfrac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \frac{1}{\cdots}}}}$$

Where a_n starts at 0, a_k is increased to infinity followed by a repeat with a_{k+1} increased by 1, starting with $k = 1$. Jumping ahead to Gatwick on the way home I find another – the integers are ordered "alphabetically", so in order of first digit, followed by second and so on, with the decimal point (these are integers so I mean the end of the integer) its own character.

The closing ceremony is brief and non-mathematical – most notably, I hold up my union flag incorrectly and we observe that the flag has rotational but not reflective symmetry, making our flag unique among those on show.

Moreover, Geoff gives a moving speech on the importance of volunteerism in the mathematical community. After the ceremony we have time for several games of pool before the closing gala. These games are all fairly mediocre, but we manage a trick shot or two.

The closing gala involves food we didn't order, music we can't stand and a conversation we can't hear, due to the ridiculous volume of the music (see Geoff's report for a more livid account). Nevertheless Matei and I manage some maths, looking at the locus of points for any triangle ABC such that the cevians through P meet opposite sides at D, E and F with $AF - FB + BD - DC + CE - EA = 0$ – the condition from the plane journey. Our expressions cannot be factorised, so we're still in limbo here. Among other loci I have been contemplating is the locus of points for all $r \in \mathbb{R}$ with perpendicular heights from each edge x, y and z that minimise $x^r + y^r + z^r$. Finally we write Salih a goodbye card, which contains various rubbish limericks such as the following (by myself):

There once was a person from Turkey,
This limerick just doesn't work-ey.
He was very cool,
But avoided the pool;
We thank you for being so perky.

Wednesday 2nd May

In which we journey home, discuss mathematics, Daniel learns German, we say our goodbyes and are reunited with our families

We wake, pack and breakfast without much maths, say our goodbyes and hop on the coach. The trip to the airport is quiet – we discuss some geometry but are generally more reflective. The airport is empty, and we move onto the plane quickly (at the gate Geoff has time to explain the role of the jury in a maths competition). On the plane, I teach Daniel some German (the flight is Airberlin so we have plenty of German to read) and continue work on our various problems. A lot of what I've described as 'solved later' throughout this report is achieved on these flights, and Matei manages to come up with a rough shape for the locus we have been looking at (for scalene triangles). I will not attempt any ASCII art to reproduce the diagram; you can do it yourself, you lazy frog. Were we a less serious bunch, our goodbyes may have been more animated, but they were moderately heartfelt nonetheless. More upsetting, as we realised upon being reunited with our parents, was the prospect of having to tidy our own rooms.

Remarks

I'd like to thank Geoff, for fighting our corner with the co-ordinators; Gerry, for keeping an eye on us and supplying us with many problems;

UKMT, for coughing up so much money; Bev, for organising everything; Tübitak, the Turkish government's science trust; the chavs from the plane, for entertaining us so greatly; the hotel, for feeding us with wonderful nourishment and providing us with such luxuries as a pool, cleaners and loo roll; and most of all, Salih the GUNK (Guide for the UNited Kingdoms) who was helpful, caring, funny, friendly and knowledgeable, and with whom I shared many a conversation consisting solely of quoting Shakespeare, Milton and Leonard Cohen.

Disclaimer: literally no effort was made whatsoever to make this report truthful in any way, shape or form. If you find a factual error, please write your correction on A4 paper along with your contact details and put it in the bin.

Appendix

Competitors from the UK

#	Name	Q1	Q2	Q3	Q4	Total	Award
1	Robin Elliott	8	10	8	1	27	Bronze
2	Gabriel Gendler	1	9	10	0	20	Bronze
3	Daniel Hu	10	0	10	10	30	Silver
4	Matthew Jasper	4	0	10	1	15	Honerable* Mention
5	Matei Mandache	10	10	9	4	33	Silver
6	Harry Metrebian	10	4	0	0	14	Honerable* Mention

*Honerable is the spelling used on the certificate

Answers

1 – Suppose, for a contradiction that such a function exists (call it f), so $f : \mathbb{R} \to \mathbb{R}\setminus\mathbb{Q}$ and $f^{-1} : \mathbb{R}\setminus\mathbb{Q} \to \mathbb{R}$. The domain of f^{-1} is the disjoint union of P and $-P$, where P is the set of positive real numbers which are not rational. Since f preserves order, so does its inverse, so the sets $f^{-1}(-P)$ and $f^{-1}(P)$ are a Dedekind cut of the reals – either $f^{-1}(-P)$ contains a maximum element or $f^{-1}(P)$ contains a minimum element, implying that either P or $-P$ has a minimum or a maximum respectively, which is absurd.

2 – the answer is not, as some might suggest, the Cantor set. While the Cantor set does cover an infinitely small interval, when it is applied over an area one third the size it's interval reduces by a half, so the guests could still feel crammed. Instead, use an interlacing – assign each bus a real number between 0 and 1 ($\frac{1}{2^n}$ will do), assign each passenger on the bus a real number between 0 and 1, write out the binary expansion and interlace the two sequences (e.g. if one begins 0.01101110... and the other 0.00011011... the end result will be 0.0010100111101101...) since just $\aleph_0$ of the first real numbers are needed, the resulting rooms never form an interval at all, so nobody feels crammed.

3 – Gerry labels D on AB such that $DT = BT$ and $B \neq D$. Now $DT = BT = CT$. DT is extended to meet AC at E. He notes that $\angle BCA = \angle TBA$ by the alternate segment theorem (note that the use of directed angles renders this true rather than that $\angle TBA + \angle BCA = 180°$ which is wrong). $\angle TBA = \angle TBD$ and $\angle TBD = \angle BDT$ since $\triangle BDT$ is

isosceles, so $\angle BCA = \angle BDT$. Therefore $\triangle ABC$ and $\triangle AED$ are similar, although mirrored, and $\angle TEC = \angle ABC$. But $\angle ABC = \angle ACT$ by the alternate segment theorem, and $\angle ACT = \angle ECT$. Since $\angle ECT = \angle TEC$, $TE = TC$. Since $TC = TB = TD$, $TE = TD$, so T is the midpoint of DE. Recall that $\triangle ABC$ is similar to $\triangle AED$, so $\triangle AMC$ is similar to $\triangle ATD$ and $\angle TAD = \angle CAM$ – QED.

4 – Harry labels point G at the intersection of AO and ℓ. M is the midpoint of AB. $2\angle ACB = \angle AOB = 2\angle AOM$ so $\angle ACB = \angle AOM$, and $\angle AMO$ is trivially 90° so $\angle AOM = 90^\circ - \angle MAO = 180^\circ - \angle DGA - \angle DAG = \angle ADG$, so $\angle BDE = \angle BCE$. Therefore $BDCE$ is concyclic, and since $\angle DCE = 90^\circ$, $\angle EBD = 180^\circ - 90^\circ = 90^\circ$. Since A, C, F and B are on Γ, $\angle ACF + \angle FBA = 180^\circ$, so $\angle FBA = 180^\circ - \angle ACF = \angle ACD - \angle ACF = \angle FCD$. Then let the midpoint of EF be N and that of FD be P. Let the circumcentres of $\angle EBF$ and $\angle FCD$ be O_1 and O_2 respectively. Then $2\angle FBE = \angle FO_1E = 2\angle FO_1N$. Similarly $2\angle FBE = 2\angle FCD = \angle FO_2D = 2\angle FO_2P$ and $FO_1N = FO_2P$. Since O_1N is perpendicular to ℓ, which in turn is perpendicular to PO_2, O_1N is parallel to O_2P and by the converse of alternate angles O_1FO_2 is collinear and we're home and dry.

5 – Robin expands out the a, b, c inequality for

$$a^2bc + ab^2c + abc^2 \geq 2a^2b^2 + 2b^2c^2 + 2a^2c^2 - a^4 - b^4 - c^4$$

Then he move the fourth powers to the LHS and requires to prove that:

$$a^2bc + ab^2c + abc^2 + a^4 + b^4 + c^4 \geq 2a^2b^2 + 2b^2c^2 + 2a^2c^2$$

Now he cites Schur's inequality ($\sum_{\text{cyc}} x^t(x-y)(x-z) \geq 0$) with t=2:

$$\sum_{\text{cyc}} a^2(a-b)(a-c) \geq 0$$

which expands out to

$$a^4 + b^4 + c^4 + a^2bc + ab^2c + abc^2 \geq \sum_{\text{sym}} a^3b$$

The LHS is the LHS of his RTP (required to prove) inequality. Now it suffices to show that:

$$\sum_{\text{sym}} a^3b \geq \sum_{\text{sym}} a^2b^2$$

Which yields trivially to Muirhead.

6 – Noting that $a + b > c$ and so on, I consider the triangle with sides of length a, b and c. The inequality at hand is

$$a^2bc + ab^2c + abc^2 \geq \sum_{\text{cyc}} (a^2 + b^2 - c^2)(a^2 + c^2 - b^2)$$

Now by the cosine law $a^2 + b^2 - c^2 = 2abcosC$, so this becomes

$$a^2bc + ab^2c + abc^2 \geq 4a^2bc\cos B\cos C + 4ab^2c\cos A\cos C + 4abc^2\cos A\cos B$$

The area of a triangle (which I call $\triangle$) formula gives $2\triangle/\sin C = ab$ and so on, so after this substitution, division by $4\triangle^2$ and multiplication by $\sin A\sin B\sin C$ I get

$$\sin A + \sin B + \sin C \geq 4\sin A\cos B\cos C + 4\sin B\cos A\cos C + 4\sin C\cos A\cos B$$

The RHS is rewritten as

$$\sum_{\text{cyc}} 2\sin A\cos B\cos C + 2\sin B\cos A\cos C = 2\sum_{\text{cyc}} \cos C(\sin A\cos B + \sin B\cos A)$$

$\sin A\cos B + \sin B\cos A = \sin(A + B)$ by the compound angle formula, and $\sin(A + B) = \sin(180 - A - B) = \sin C$, so the RHS is just

$$\sum_{\text{cyc}} 2\sin A\cos A$$

This is the double angle formula, so I am required to prove:

$$\sin A + \sin B + \sin C \geq \sin 2A + \sin 2B + \sin 2C$$

Now I recall that $\triangle ABC$ is acute, so for any $A \geq B$, $2\sin A \geq 2\sin B$ and $\cos A \leq \cos B$. Hence by the rearrangement inequality

$$2(\sin A\cos B + \sin B\cos A) \geq 2\sin A\cos A + 2\sin B\cos B$$

$$2\sin C \geq \sin 2A + \sin 2B$$

Taking the sum of this inequality cyclically and dividing by 2 gives

$$\sin A + \sin B + \sin C \geq \sin 2A + \sin 2B + \sin 2C$$

as required.

7 – $x + y \geq 2\sqrt{xy}$ (by simple AM-GM)

$$(x+y)z \geq 2z\sqrt{xy}$$

(multiplying by z)

$$z^2 + (x+y)z + xy \geq z^2 + 2z\sqrt{xy} + xy$$

(adding $z^2 + xy$)

$$(z+x)(z+y) \geq (z+\sqrt{xy})^2$$

(factorising)

$$\sqrt{z+x}\sqrt{z+y} \geq z + \sqrt{xy}$$

(rooting)

$$(x+y)\sqrt{z+x}\sqrt{z+y} \geq (x+y)(z+\sqrt{xy})$$

(multiplying by (x+y)).

This applies cyclically, so:

$$\text{LHS} \geq \sum_{\text{cyc}} (x+y)(z+\sqrt{xy})$$

$$\text{LHS} \geq 2\sum_{\text{cyc}} xy + \sum_{\text{cyc}} (x+y)\sqrt{xy}$$

Since $x + y \geq \sqrt{xy}$ by AM-GM

$$\text{LHS} \geq 2\sum_{\text{cyc}} xy + 2\sum_{\text{cyc}} \sqrt{xy}\sqrt{xy}$$

$$\text{LHS} \geq 4\sum_{\text{cyc}} xy$$

LHS $\geq$ RHS as required.

8 – Daniel starts with the standard a, b, c substitution, then expands out the RHS, factorises the LHS and requires:

$$(a+b+c)abc = 2\sum_{\text{cyc}} a^2b^2 - \sum_{\text{cyc}} a^4$$

like Robin. The RHS factorises to $(a+b+c)(a+b-c)(a+c-b)(b+c-a)$, so for a triangle with sidelengths a, b and c,

$$2abc(a+b+c)/2 \geq (a+b+c)(a+b-c)(a+c-b)(b+c-a)$$

The RHS is Heron's formula squared multiplied by 16, so

$$2abc(a+b+c)/2 \geq 16\triangle^2$$

We write one $\triangle$ as $\frac{abc}{4R}$ and the other as rs, so

$$2abcs \geq 16\frac{abcrs}{4R}$$

Which simplifies to
$R \geq 2r$, Euler's inequality.

9 – Matthew will prove by induction, first showing that for any given n, $n+1$ also works and then looking at the base case when $n = 1$.
P_n is good from a to b if:
$\forall a \leq y \leq b \quad \exists Y \subseteq P_n$ such that $0 \leq y - S_Y \leq 2^n$
If P_n is good from 0 to $3^{n+1} - 2^{n+1}$ then it's great.

Claim 1: $\forall P_n \quad S_{P_n} = 3^{n+1} - 2^{n+1}$

Proof: I did this by induction but since I did MPC2 yesterday and I like geometric series:

$S_{P_n} = a\frac{1-r^{n+1}}{1-r} = 2^n \frac{1-\frac{3}{2}^{n+1}}{1-\frac{3}{2}} = 3^{n+1} - 2^{n+1}.\Box$

Claim 2: $\forall$ great P_n , $\exists \quad U_0, U_1, U_2, \ldots, U_m \quad U_i \subseteq P_n$ such that S_{U_i} are increasing $U_0 = \emptyset$, $U_m = P_n$ and $S_{U_j} - S_{U_{j-1}} \leq 2^n \quad \forall 0 < j \leq m.$

Proof: Only the final condition needs to be proved as all can be made to be true or have been shown before. Suppose $y = S_{U_j} - 1$ then $y - S_{U_{j-1}} < 2^n$ and $S_{U_j} - 1 - S_{U_{j-1}} < 2^n$.

LHS is an integer so $S_{U_j} - S_{U_{j-1}} < 2^n.\Box$

Claim 3: This is also a sufficient condition for P_n being great

Proof: Let $S_{U_{j-1}} \leq y < S_{U_j}$, which can be true for all relevant y. Subtract $S_{U_{j-1}}$: $0 < y - S_{U_{j-1}} < S_{U_j} - S_{U_{j-1}}$ and it follows that $0 \leq y - S_{U_{j-1}} \leq 2^n.\Box$

Claim 4: P_{n-1} is great implies P_n is great. $n \geq 2$

Proof: Let $X_0, X_1, \ldots X_m$ be the list of subsets describe above for P_{n-1}. Let $V_0, V_1, \ldots V_m$ be subsets of P_n with elements twice that of corresponding X and let $W_0, W_1, \ldots W_m$ be the same subsets but also containing 3^n.

Now:

$9 \cdot 3^{n-2} - 8 \cdot 2^{n-2} > 0 (n \geq 2)$

$2 \cdot (9 \cdot 3^{n-2}) - 8 \cdot 2^{n-2} > 9 \cdot 3^{n-2}$

$2 \cdot (3^n - 2^n) > 3^n$

$2 \cdot (S_{X_m}) > 3^n P$

$S_{V_m} > S_{W_0}$

This means the set containing all V_i and W_i is suitable to show P_n is good because differences of sums is twice that of X_i which is suitable and the two sets have sums covering the whole range of y and overlap so this makes P_n great.$\Box$

Finally note P_1 is great:

$\emptyset, \{2\}, \{3\}, \{2, 3\} \subseteq P_n$

and give values of S_Y 0,2,3,5. which shows P_1 is great $(0 \leq y \leq 3^2 - 2^2 = 5)$ and by induction all P_n are great.$\Box$

10 – Daniel starts by supposing that the function is eventually constant. Then $f(n) = k$ for all $n \geq n_0$, for some constant k.

We prove by downwards induction that $f(n) = k$ for all n. If $n_0 > 1$ then $n_0 \in \mathbb{Z}^+$. If $n_0 - 1 \geq 3$, then $k = f((n_0 - 1)!) = f(n_0 - 1)!$ as $n_0 - 1 > n_0$ for all $n_0 - 1 \geq 3$. We also have $f(n_0!) = k$ since $n! \geq n \; \forall \; n$, so $f(n_0)! = k$. Moreover $f(n_0) = k$ so $f(n_0)! = k!$, so $k! = k$. We have $k = f(n_0 - 1)!$, so $k! = f(n_0 - 1)!$, $k = f(n_0 - 1)$ which contradicts the definition that n_0 is the lowest number such that $f(n) = k \; \forall \; n \geq n_0$. Therefore $n_0 - 1 \geq 3$ is impossible, so $n_0 - 1 < 3$, so $n_0 - 1$ is 1 or 2. Therefore $(n_0 - 1)! = n_0 - 1$, so $f(n_0 - 1)! = f(n_0 - 1)$ so $f(n_0 - 1)$ is 1 or 2.

By the second rule, $f(n_0+1) - f(n_0-1)$ is a multiple of 2. $f(n_0+1) = k$, so $k \equiv f(n_0 - 1) \pmod 2$. Moreover both k and $f(n_0 - 1)$ are 1 or 2, and $1 \neq 2 \pmod 2$, $k = f(n_0 - 1)$. Therefore the assumption that $n_0 > 1$ was wrong, but $n_0 \geq 1$ so $n_0 = 1$ – in other words, if the function is eventually constant it must be constant throughout. $f(n) = k \; \forall \; n$ where k is 1 or 2. Both are trivially solutions, so we have all such functions where $f(n)$ is eventually constant.

If f is not eventually constant, the $\forall \; n, \exists k \in \mathbb{Z}^+$ such that $f(n+k) \neq f(n)$. We have

$$\frac{f((n+k)!) - f(n!)}{(n+k)! - n!} = \frac{f(n+k)! - f(n)!}{(n+k)! - n!} \in \mathbb{Z}$$

Suppose for contradiction that $f(n) < n$ for some n. However, $n < n+k$ so $a \cdot n! = (n+k)!$ for some $a \geq n+1$. Therefore $(a-1)n! = (n+k)! - n!$, so $n! \mid (n+k)! - n!$. $(n+k)! - n! \mid f(n+k)! - f(n)!$ by the second rule so $n! \mid f(n+k)! - f(n)!$. $f(n+k)! - f(n)! \neq 0$ since $f(n+k) \neq f(n)$ by definition. Therefore either $f(n+k)! - f(n)! \geq n!$ or it is negative.

If $f(n+k)! - f(n)!$ is negative, it is at least $-f(n)!$ since $f(n+k)!$ is positive. Moreover it is at most $-n!$ since it divides $n!$, so $-f(n)! \leq -n!$, which implies that $n! \leq f(n)!$, so $n \leq f(n)$ which contradicts our definition. Hence $f(n+k)! - f(n)! \geq n!$, so $f(n+k)! > n!$, so $f(n+k) > n$. Therefore $n! \mid f(n+k)!$, but also $n! \mid f(n+k)! - f(n)!$, so $n! \mid f(n)!$ which implies that $n \leq f(n)$ – contradiction.

Therefore $f(n) \geq n \; \forall \; n$. As we saw in the case where $f(n)$ is eventually constant, $f(2) = f(2!) = f(2)!$ so $f(2)$ is its own factorial, so $f(2)$ is 1 or 2. However since $f(2) \geq 2$, $f(2) = 2$.

Now suppose there exists $n \geq 3$ such that $f(n) \geq n! - 2$. Since

$$\frac{f(n!) - f(2!)}{n! - 2!} \in \mathbb{Z}$$

$$\frac{f(n!) - 2}{n! - 2} \in \mathbb{Z}.$$

$n! - 2 \leq f(n)$ so $n! - 2 \mid f(n)!$, which implies from the two divisibilities that $n! - 2 \mid 2$. However, $n > 3$, so $n! - 2 > 2$, contradiction. Therefore $\forall\, n \geq 3$, $f(n) \leq n! - 3$ – so $f(3) \leq 3! - 3 = 3$. $f(3) \geq 3$, so $f(3) = 3$.

$f(1) = f(1!) = f(1)!$ so $f(1)$ is 1 or 2, but by the second rule $f(3) \equiv f(1) \pmod 2$, so $f(1) \equiv 1 \pmod 2$ and $f(1) = 1$.

So we have:

$f(1) = 1$

$f(2) = 2$

$f(3) = 3$

$\forall\, n, n \mid n! - n$, and $n! - n \mid f(n!) - f(n)$, so $n \mid f(n)! - f(n)$. Now $f(n) \geq n$, so $n \mid f(n)!$. Hence $n \mid f(n)$.

We now use induction to show that $f(n) = n \;\forall\, n$. We know this is true when n is 1, 2 and 3 as our base cases. We suppose that $f(n) = n \;\forall\, n \leq k$ for some $k \geq 3$. Then

$$\frac{f(k+1)! - f(k!)}{(k+1)! - k!} \in \mathbb{Z}$$

Hence

$$\frac{f(k+1)! - k!}{(k+1)! - k!} \in \mathbb{Z}$$

since $f(k!) = f(k)! = k!$. The denominator is $(k+1)! - k! = (k+1)k! - k! = k(k!)$, so

$$\frac{f(k+1)! - k!}{k(k!)} \in \mathbb{Z}$$

Suppose for a contradiction that $f(k+1) \geq 2k$ – then $(2k)! \mid f(k+1)!$ and as $k \mid 2k$ and $k! \mid (2k-1)!$, $k(k!) \mid 2k(2k-1)!$, so $k(k!) \mid 2k!$, and $k(k!) \mid f(k+1)!$. Therefore from the two divisibilities, $k(k!) \mid k!$ but $k(k!) > k! > 0$, so this is a contradiction. Therefore $f(k+1) < 2k$. $k+1 \mid f(k+1)$ from above, so $k+1 \leq f(k+1) < 2k$, and the only multiple of $k+1$ in this interval is $k+1$. Hence $k+1 = f(k+1)$ and induction is complete. Therefore the only f which suits the conditions and which is not eventually constant is $f(n) = n$. This works trivially, so the three solutions are $f(n) = 1$, $f(n) = 2$ and $f(n) = n$.

11 – Invert the plane about A, with radius AQ. It is not hard to show that Γ' is the line PQ, and vice versa. Hence B' is P and P' is B. The new

circle must also be in exactly the same place, so it is tangential to $\infty' B$ at G', which is also G. Hence $G = G'$ and $AG = AQ$.

12 – we start with $2R = \frac{a}{\sin A}$. Multiplying by $\sin A$ and adding up cyclically gives:

$$2R(\sin A + \sin B + \sin C) = a + b + c$$

Then multiplying by $\frac{r}{2}$

$$Rr(\sin A + \sin B + \sin C) = rs = \triangle$$

$\triangle = \frac{R^2}{2}(\sin 2A + \sin 2B + \sin 2C)$, so:

$$Rr(\sin A + \sin B + \sin C) = \frac{R^2}{2}(\sin 2A + \sin 2B + \sin 2C)$$

$$r(\sin A + \sin B + \sin C) = \frac{R}{2}(\sin 2A + \sin 2B + \sin 2C)$$

Euler tells us that $r \leq \frac{R}{2}$, so

$$r(\sin A + \sin B + \sin C) = \frac{R}{2}(\sin 2A + \sin 2B + \sin 2C) \geq r(\sin 2A + \sin 2B + \sin 2C)$$

$$\sin A + \sin B + \sin C \geq \sin 2A + \sin 2B + \sin 2C$$

and in the words of the great József Pelikán, we are ready.

13 – take any five points forming a convex pentagon, and consider the set of points inside the pentagon, include the 5 that form the convex hull. Let the number of black points be b and the number of green points be g. The black points form a b-gon, which can be triangulated into $b-2$ triangles, so there are at least $b-2$ green points inside of the convex hull. Similarly there are at least $g-2$ blue points inside of the convex hull, and a total of at least $b+g-4$ points inside of the convex hull. Including the convex hull of 5, there are at least $b+g+1$ points, but by definition there are $b+g$ – contradiction.

14 – taking everything $\log_{1/4}$ reverses the inequality, since it is a decreasing function. This demonstrates the fact that multiplication by -1 and finding the reciprocal are not the only functions that reverse the sign – they are just two cases of the general rule that applying a decreasing function changes the sign. This fact, while intuitive, seems to be skipped in schools.

15 – I start with a mental catechism (really, it helps).

Me: Why is this problem difficult?

Me: It doesn't allow for heavy machinery such as Muirhead's.

Me: Why doesn't it respond to Muirhead's?

Me: Because different terms have different orders – the expression is not homogeneous.

Me: How can these orders be adjusted?

Me: If an identity in x, y and z exists with different orders on the two sides.

Me: Does such an expression exist?

Me: No, because $x + y + z = 0$.

Me: Wait a second – if $x+y+z = a$ for some non-zero constant a, we're sorted, right?

Me: Right – we could multiply everything with lower order by $(x+y+z)/a$ repeatedly, increasing the order indefinitely.

Me: Great!

Me: But $x + y + z = 0$.

Me: But $(x + 1) + (y + 1) + (z + 1) = 3\ldots$

Me: Oh, I see – we substitute in $t = x + 1$, $u = y + 1$ and $v = z + 1$ and we can make it homogeneous.

Me: And then we can bash it!

Me: The expansion would involve over 900 terms.

Me: Fine, but at least I have a way of solving it.

Me: You don't know it will work.

Me: Five pounds if it works.

Me: Who does the expansion?

Me: Nobody needs to do the expansion.

Me: Why not?

Me: Because we have the same wallet, you and I, so whoever is wrong may as well have just transferred the money.

Me: Well, whichever way it is, I just picked up a fiver, so I'm going to Tesco to buy a celebratory chocolate bar with my hard earned cash.

16 – Daniel takes us halfway by identifying several similar triangles found by drawing cevians through P and hence finding an expression that doesn't look far off what we're required to prove – that

$$\frac{1}{\sin^2 A} + \frac{1}{\sin^2 B} + \frac{1}{\sin^2 C} = \frac{1}{\sin^2 \omega}$$

The reciprocal of the sine is the cosecant, and the square of the cosecant is 1 more than the cotangent, so we have

$$\cot^2 A + \cot^2 B + \cot^2 C = \cot^2 \omega - 2$$

Bear this equation in mind for a moment. We know that $\tan A + \tan B + \tan C = \tan A \tan B \tan C$; dividing this by its own RHS and multiplying by two gives

$$2 \sum_{\text{cyc}} \cot A \cot B = 2$$

Adding this equation to the earlier one gives

$$\sum_{\text{cyc}} \cot^2 A + 2 \sum_{\text{cyc}} \cot A \cot B = \cot^2 \omega$$

Square rooting gives the desired inequality.
Jensen's (along with some differentiation) gives the corollary.

17 – Multiply away the denominators and expand to require

$$\sum_{\text{cyc}} a^3 b^3 - \sum_{\text{cyc}} a^3 b^2 c \geq 0$$

$$\sum_{\text{cyc}} a^3 b^3 \geq \sum_{\text{cyc}} a^3 b^2 c$$

Which yields to rearrangement with sets $\{a^2b^2, a^2c^2, b^2c^2\}$ and $\{ab, ac, bc\}$.

18 – Dividing by abc, Daniel requires

$$\sum_{\text{cyc}} \frac{a(b-c)}{c(a+b)} \geq 0$$

He now adds 3. Wow.

$$\sum_{\text{cyc}} \frac{a(b-c)}{c(a+b)} + 1 \geq 3$$

$$\sum_{\text{cyc}} \frac{b(a+c)}{c(a+b)} \geq 3$$

Which is AM-GM since the terms multiply to 1.