UK IMO team leader's report

Geoff Smith, University of Bath

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This year the International Mathematical Olympiad was held in Rio de Janeiro. The organizers did an excellent job, and the event ran smoothly throughout. The relaxed informal Brazilian style is infectious.

The IMO is the world championship of secondary school mathematics, and is held each July in a host country somewhere in the world. A modern IMO involves more than 110 countries, representing over 90% of the world's population. The competition was founded in 1959. Each participating country may send up to six team members, who must be under 20 years of age and not have entered university.

The UK Deputy Leader was Dominic Yeo of the Technion, and our Observer C was Jill Parker, formerly of the University of Bath. Here is the UK IMO team of 2017:

Joe Benton	St Paul's School, Barnes, London
Rosie Cates	Hills Road Sixth Form College, Cambridge
Jacob Coxon	Magdalen College School, Oxford
Neel Nanda	Latymer School, Edmonton, London
Alexander Song	Westminster School, London
Harvey Yau	Ysgol Dyffryn Taf, Carmarthenshire, Wales

The reserves were Sam Bealing, Bridgewater High School; Michael Ng, Aylesbury Grammar School; Thomas Read, The Perse School; Naomi Wei, City of London School for Girls.

Here are the results obtained by the UK students this year.

Name	P1	P2	P3	P4	P5	P6	Σ	award
Joe Benton	7	7	5	7	1	2	29	Gold
Rosie Cates	7	1	0	7	0	3	18	Bronze
Jacob Coxon	7	3	0	7	0	0	17	Bronze
Neel Nanda	7	4	0	7	0	7	25	Gold
Alexander Song	7	1	0	7	0	0	15	Honourable Mention
Harvey Yau	7	1	0	7	7	4	26	Gold

There are three problems to address on each of two consecutive days. Each exam lasts 4 hours 30 minutes. The cut-offs were 16 for bronze, 19 for silver and 25 for gold. The current IMO marks format became stable in 1982. This is the lowest gold cut, and the equal lowest silver cut, since then. This is evidence of the exceptional difficulty of this IMO.

This is the first time that a UK team has secured three gold medals at an IMO since 1981, the last year in which IMO teams comprised 8 people. Joe, Harvey and Neel all obtained well deserved gold medals.

It was very pleasing that EGMO star (41/42) Rosie Cates made the team, and secured a strong bronze medal. Jacob Coxon also earned a good bronze. Alexander Song was unlucky to miss out on a medal by 1 mark, but he has chances to win medals at IMOs 2018 and 2019. His marks were essential in securing a high team ranking for the UK. Harvey is also available for selection for IMO 2018.

Performances

The joint winner of IMO 2017 was Yuta Takaya of Japan who scored 35/42, securing perfect scores on all problems which did not involve rabbits. Immediately after IMO 2017, he participated in the International Olympiad in Informatics in Iran, and he won that too. Breathing closely down Yuta's neck was the UK's Joe Benton who came 7th in the IMO and 6th at the IOI. Joe was able to say very worthwhile things about invisible leporidae, producing a solution (with correct asymptotic analysis) but with details of the calculation missing because of shortage of time.

There were 111 teams participating at IMO 2017. Hearty congratulations to Korea for finishing ranked 1st, repeating their first win of 2012. All six Korean students (including a girl) won gold medals, and no other team repeated this feat in 2017. We give a list of the teams ranked in the top 40, and by chance this list consists of exactly those teams which scored at least 100 marks.

1 Korea (170), 2 China (159), 3 Vietnam (155), 4 USA (148), 5 Iran (142), 6 Japan (134), 7 Singapore, Thailand (131), 9 Taiwan, United Kingdom (130), 11 Russian Federation (128), 12 Georgia, Greece (127), 14 Belarus, Czech Republic, Ukraine (122), 17 Philippines (120), 18 Bulgaria, Italy, Netherlands, Serbia (116), 22 Hungary, Poland, Romania (115), 25 Kazakhstan (113), 26 Argentina, Bangladesh, Hong Kong (111), 29 Canada (110), 30 Peru (109), 31 Indonesia (108), 32 Israel (107), 33 Germany (106), 34 Australia (103), 35 Croatia, Turkey (102), 37 Brazil, Malaysia (101), 39 France, Saudi Arabia (100)

Anglophone and Commonwealth interest in other scores might include 46 New Zealand (94), 47 Cyprus (93), 52 India (90), 60 South Africa (81), 62 Ireland, Sri Lanka (80), 81 Pakistan (58), 95 Uganda (22), 98 Botswana (19), 99 Myanmar, Trinidad and Tobago [1 person] (15), 107 Kenya (8), 108 Ghana [1 person] (6), 109 Tanzania [2 people] (5), 110 Nepal (3).

China finished 2nd, continuing their extraordinary run of consistently excellent performances. China has not finished outside the top 3 since 1996. As in 2016, the top 10 countries comprise 7 countries from the far east, and three others. In 2016 the three exceptions were Russia, UK and USA, and in 2017 Iran replaced

Russia in that trio.

Top monarchy was Japan, and Commonwealth Champion was Singapore. The UK was first among the nations of Europe, and therefore also those of the EU. Note the remarkable performances of Greece and Georgia in finishing equal 12th, so equal third in Europe just behind Russia. Greece was first among nations which use the euro.

From our national perspective, the breadth of the UK performance was very pleasing. Every team has a virtual player Max. This player is deemed to score the maximum of the marks obtained by a team member on each problem. In the Max contest, the clear winner was UK Max who scored 40/42, dropping just two marks on Problem 3. Second equal were Korean Max and Russian Maxim who scored 36/42.

Full scores are available at the IMO official site:

`https://www.imo-official.org/year_country_r.aspx?year=2017`

The Papers

Contestants have 4 hours 30 minutes to sit each paper. The three problems on each paper are each marked out of 7. It is intended that the three problems should be in increasing order of difficulty on each day.

Day 1

1. For each integer $a_0 > 1$, define the sequence $a_0, a_1, a_2, \ldots$ by:

$$a_{n+1} = \begin{cases} \sqrt{a_n} & \text{if } \sqrt{a_n} \text{ is an integer,} \\ a_n + 3 & \text{otherwise,} \end{cases} \qquad \text{for each } n \geqslant 0.$$

   Determine all values of a_0 for which there is a number A such that $a_n = A$ for infinitely many values of n.

2. Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that, for all real numbers x and y,

$$f\left(f(x)f(y)\right) + f(x+y) = f(xy).$$

3. A hunter and an invisible rabbit play a game in the Euclidean plane. The rabbit's starting point, A_0, and the hunter's starting point, B_0, are the same. After $n-1$ rounds of the game, the rabbit is at point A_{n-1} and the hunter is at point B_{n-1}. In the n^{th} round of the game, three things occur in order.

   (i) The rabbit moves invisibly to a point A_n such that the distance between A_{n-1} and A_n is exactly 1.

   (ii) A tracking device reports a point P_n to the hunter. The only guarantee provided by the tracking device to the hunter is that the distance between P_n and A_n is at most 1.

(iii) The hunter moves visibly to a point B_n such that the distance between B_{n-1} and B_n is exactly 1.

Is it always possible, no matter how the rabbit moves, and no matter what points are reported by the tracking device, for the hunter to choose her moves so that after 10^9 rounds she can ensure that the distance between her and the rabbit is at most 100?

Day 2

4 Let R and S be different points on a circle Ω such that RS is not a diameter. Let ℓ be the tangent line to Ω at R. Point T is such that S is the midpoint of the line segment RT. Point J is chosen on the shorter arc RS of Ω so that the circumcircle Γ of triangle JST intersects ℓ at two distinct points. Let A be the common point of Γ and ℓ that is closer to R. Line AJ meets Ω again at K. Prove that the line KT is tangent to Γ.

5 An integer $N \geqslant 2$ is given. A collection of $N(N+1)$ soccer players, no two of whom are of the same height, stand in a row. Sir Alex wants to remove $N(N-1)$ players from this row leaving a new row of $2N$ players in which the following N conditions hold:

(1) no one stands between the two tallest players,

(2) no one stands between the third and fourth tallest players,

$\vdots$

(N) no one stands between the two shortest players.

Show that this is always possible.

6 An ordered pair (x, y) of integers is a *primitive point* if the greatest common divisor of x and y is 1. Given a finite set S of primitive points, prove that there exist a positive integer n and integers a_0, a_1, ..., a_n such that, for each (x, y) in S, we have:

$$a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \cdots + a_{n-1}xy^{n-1} + a_ny^n = 1.$$

Problem Authors

Problem 1 was proposed by South Africa (Stephan Wagner),
Problem 2 was proposed by Albania (Dorlir Ahmeti),
Problem 3 was proposed by Austria (Gerhard Woeginger),
Problem 4 was proposed by Luxembourg (Charles Leytem),
Problem 5 was proposed by Russia (Grigory Chelnokov),
Problem 6 was proposed by the United States of America (John Berman).

Comments on Problems

It was pleasing that the UK team obtained perfect scores on the first problem of each day. Certainly Problem 1 was a completely appropriate choice, and gave lots of students the opportunity to obtain an Honourable Mention. The geometry Problem 4 was amenable to attack by sensible methods, but naturally Harvey Yau found another approach. He began by discussing some barely relevant spiral similarities, and then created an extraordinary collection of circles with curious intersection properties, and concluded by means of a collinearity which follows from the converse of Miquel and Steiner's quadrilateral theorem. Under no circumstances should you try this at home.

The two medium problems, 2 and 5, proved hard this year. Problem 2 was a functional equation which demanded considerable ingenuity. Problem 5 was an *hommage* to the Erdős-Szekeres theorem which, in downmarket form, concerns soccer players of different heights standing in a row. Problem 5 was one of those conceptual combinatorial problems which students find hard to solve under pressure of time.

The two hard problems were indeed hard, especially problem 3. Again this was a conceptual problem, where the technical mathematics was not demanding but the idea of how to solve the problem was very hard to find. Only two students in the IMO scored perfectly on Problem 3 (Linus Cooper of Australia and Mikhail Ivanov of Russia) and only 26 marks were awarded for that problem in total. Problem 6 was an intriguing and beautiful problem about homogeneous polynomials.

Forthcoming International Events

This is a summary of the events which are relevant for the UK. Of course there are many other competitions going on in other parts of the world.

The next IMO will be held in Cluj, Romania 2018. More formally, this Transylvanian city is known as Cluj-Napoca. After that the IMOs are: Bath, UK 2019; St Petersburg, Russia 2020; the USA 2021, Norway 2022 and (unconfirmed) Japan 2023. Forthcoming editions of the European Girls' Mathematical Olympiad will be in Florence, Italy in 2018; Kiev, Ukraine in 2019; Netherlands 2020 and (unconfirmed) Georgia 2021. The Balkan Mathematical Olympiad of 2018 will be in Serbia. The Romanian Master of Mathematics competition will continue as an annual event in Bucharest.

The Mathematical Ashes

The United Kingdom retained the Ashes $83-63$ in the 2017 match at the pre-IMO camp in Itaipava, Brazil. The UK has held the Ashes since 2009, but the contest of 2018 may see a change of fortunes, with the UK losing four IMO 2017 medallists to higher education.

Acknowledgements

Thanks to everyone in Brazil for making this such an enjoyable IMO, especially the taxi drivers who always sought to minimize journey times. In the land of Ayrton Senna, Nelson Piquet and Emerson Fittipaldi, passengers may wish to use a blindfold.

The UK Mathematics Trust is an astonishing organization, bringing together so many volunteers and a small professional core to focus their energies on maths competitions and more generally, mathematics enrichment. Our collective effort is, I am sure, a significant part of the success story which is secondary school mathematics for able students in the UK. This is not to be complacent, because there are always opportunities do more things and to do things better, but I thank everyone for what we already accomplish every year. Hundreds of thousands of lives are touched by our wonderful maths challenges and team competitions, and I thank everyone involved for their marvellous work.

On a personal note, I thank Dominic Yeo for his exceptional work, inspiring so many young people with his passion for good mathematics. Jill Parker kept the team happy and safe while it was in Brazil. The teams which UKMT sends abroad to represent the country (and the associated reserves) continue to conduct themselves in an exemplary fashion in person, in *the Guardian* and on *Radio 4*.

Joseph Myers did splendid work as a co-ordinator of Problem 3 and minute taker for both the IMO Advisory Board and the IMO jury. We were also joined for a few days by Steve Mulligan of the Team Maths Challenge subtrust who came to Rio to promote *Diamond Maths Challenge*, the global outreach project of IMO 2019 in the United Kingdom. Steve made a lot of friends very quickly.

I thank *Oxford Asset Management* for their continuing generous sponsorship of the UK IMO team, and the other donors, both individual and corporate, who give so generously to UKMT. Why not join in?

`http://www.ukmt.org.uk/about-us/`

`G.C.Smith@bath.ac.uk` `@GeoffBath`