

RMM 2017 - UK Report

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Introduction

The ninth edition of the Romanian Master of Mathematics was held in Bucharest between 22nd and 27th February 2017. This is a hard competition for school-aged mathematicians from invited countries. The UK has been delighted to accept invitations to attend each edition since the first in 2008. Our participation is arranged by the UK Maths Trust², as part of a broader programme to introduce the country's most enthusiastic young mathematicians to regular problem-solving, challenging mathematics, and several annual opportunities to participate in competitions.

This UK team was selected based on tests at our winter olympiad camp in Tata, Hungary, and corresponded for weekly problem-solving in the month before the competition.

Joe Benton	St. Paul's School, London	(18)
Rosie Cates	Hills Road Sixth Form College	(17)
Neel Nanda	Latymer School	(18)
Thomas Read	The Perse School	(18)
Alexander Song	Westminster School	(16)
Harvey Yau	Ysgol Dyffryn Taf	(17)

The reserves were Nathan Creighton (Mossbourne Community Academy, 16) and Benedict Randall Shaw (Westminster School, 14), who also participated in the correspondence training. James Gazet of Eton College was deputy leader. Mary-Teresa Fyfe, formerly of Hutcheson's Grammar School, accompanied the contestants.

Problems of the contest

This contest follows the same structure as the International Mathematical Olympiad. On two consecutive days the students sit exams lasting 4.5 hours. Each exam has three questions, and each question receives a mark out of seven.

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²<https://www.ukmt.org.uk/>

Day One

Problem 1. (a) Prove that every positive integer n can be written uniquely in the form

$$n = \sum_{j=1}^{2k+1} (-1)^{j-1} 2^{m_j},$$

where $k \geq 0$ and $0 \leq m_1 < m_2 < \dots < m_{2k+1}$ are integers. This number k is called the *weight* of n .

(b) Find (in closed form) the difference between the number of positive integers at most 2^{2017} with even weight and the number of positive integers at most 2^{2017} with odd weight.

VJEKOSLAV KOVAČ, CROATIA

Problem 2. Determine all positive integers n satisfying the following condition: for every monic polynomial P of degree at most n with integer coefficients, there exists a positive integer $k \leq n$, and $k + 1$ distinct integers x_1, x_2, \dots, x_{k+1} such that

$$P(x_1) + P(x_2) + \dots + P(x_k) = P(x_{k+1}).$$

Note. A polynomial is *monic* if the coefficient of the highest power is one.

S. PETROV, RUSSIA

Problem 3. Let n be an integer greater than 1 and let X be an n -element set. A non-empty collection of subsets A_1, \dots, A_k of X is *tight* if the union $A_1 \cup \dots \cup A_k$ is a proper subset of X and no element of X lies in exactly one of the A_i s. Find the largest cardinality of a collection of proper non-empty subsets of X , no non-empty subcollection of which is tight.

Note. A subset A of X is *proper* if $A \neq X$. The sets in a collection are assumed to be distinct. The whole collection is assumed to be a subcollection.

A. POLYANSKY, RUSSIA

Day Two

Problem 4. In the Cartesian plane, let \mathcal{G}_1 and \mathcal{G}_2 be the graphs of the quadratic functions $f_1(x) = p_1x^2 + q_1x + r_1$ and $f_2(x) = p_2x^2 + q_2x + r_2$, where $p_1 > 0 > p_2$. The graphs \mathcal{G}_1 and \mathcal{G}_2 cross at distinct points A and B . The four tangents to \mathcal{G}_1 and \mathcal{G}_2 at A and B form a convex quadrilateral which has an inscribed circle. Prove that the graphs \mathcal{G}_1 and \mathcal{G}_2 have the same axis of symmetry.

A. ZASLAVSKY, RUSSIA

Problem 5. Fix an integer $n \geq 2$. An $n \times n$ *sieve* is an $n \times n$ array with n cells removed so that exactly one cell is removed from every row and every column. A *stick* is a $1 \times k$ or $k \times 1$ array for any positive integer k . For any sieve A , let $m(A)$ be the minimal number of sticks required to partition A . Find all possible values of $m(A)$, as A varies over all possible $n \times n$ sieves.

PALMER MEBANE AND NIKOLAI BELUHOV

Problem 6. Let $ABCD$ be any convex quadrilateral and let P, Q, R, S be points on the segments AB, BC, CD , and DA , respectively. It is given that the segments PR and QS dissect $ABCD$ into four quadrilaterals, each of which has perpendicular diagonals. Show that the points P, Q, R, S are concyclic.

NIKOLAI BELUHOV

Discussion of the problems

The following commentaries on each problem are not supposed to be official solutions, though some do include solutions, or substantial steps of solutions. As such, anyone hoping to try the problems themselves (especially any potential olympiad contestants) would be advised to postpone reading this section, though the first half of each commentary might be a useful if substantial hint.

At this competition, the leaders have a small amount of time to think about the problems before they are confirmed for the paper. As a result it is hard to become expert on all six during the timeframe of the competition. Obviously, the contestants also spend potentially several hours attempting these problems, and I see no reason why their views should be less relevant than mine. So the accounts of Problems 1 and 3 are written by UK students, partly based on their solutions during the contest itself. In the remainder, I have tried to

emphasise some key ideas and general principles, and how one might have arrived at them naturally, though both stages of this are highly subjective.

Problem 1

By Rosie Cates and Neel Nanda

a) We are trying to express n in terms of powers of 2, so it seems sensible to write n in binary. As 2^{m_1} is the smallest power of 2, this term is responsible for the last 1 in the binary representation of n . Let $x = n - 2^{m_1}$ (ie n with the last 1 removed from its binary expansion). Now if we pair up terms in the sum to get

$$x = (2^{m_{2k+1}} - 2^{m_{2k}}) + \dots + (2^{m_3} - 2^{m_2}),$$

we can see that each bracket looks like $11\dots100\dots0$ when written in binary. Also, the condition that $m_i < m_{i+1}$ is equivalent to ensuring that we do not break any strings of consecutive 1s that were in the binary expansion of x (so for example $111110 = 110000 + 1110$ is not allowed). So writing x in the desired form is the same as writing it as the sum of numbers of the form $11\dots100\dots0$ without breaking any strings of 1s. For example

$$1110100110 = 1110000000 + 100000 + 110.$$

Clearly there is exactly one way of doing this for every x , so (as each n has exactly one x) there is exactly one way to do it for each n as well.

This approach allows k to be understood differently. Write n in binary and remove the last 1; now count the number of groups of consecutive 1s. This is equal to k .

b) The second half of the problem becomes a lot simpler with the observation that $n \leq 2^{m_{2k+1}}$, as

$$n = 2^{m_{2k+1}} - (2^{m_{2k}} - 2^{m_{2k-1}}) - \dots - (2^{m_2} - 2^{m_1}),$$

and the sequence m_n is increasing, so each bracket is positive. As each sequence of (m_n) s corresponds uniquely to an integer, this means we just want to count sequences of (m_n) s with greatest term at most 2017. The sequence is increasing, so each sequence corresponds to a subset of $\{0, 1, \dots, 2017\}$ of size $2k + 1$. There are $\binom{2018}{2k+1}$ subsets of size $2k + 1$, so the question reduces to finding a closed form for $\sum_{k=0}^{1008} (-1)^k \binom{2018}{2k+1}$.

This is reminiscent of a classic problem in combinatorics: using the binomial theorem to evaluate sums of binomial coefficients weighted by powers. The best example is

$$\sum_{k=0}^n (-1)^k \binom{n}{k} = (1 - 1)^n = 0,$$

but here rather than (-1) we want something whose square is (-1) , so we consider the complex number i . Using the same ideas, we get that

$$\sum_{r=0}^{2018} i^r \binom{2018}{r} = (1+i)^{2018},$$

which contains what we want, but also binomial coefficients with even r . But if r is even, i^r is real, and if r is odd, i^r is imaginary. So the sum we want appears as the imaginary part, that is

$$\operatorname{Im}((1+i)^{2018}) = \operatorname{Im}\left((\sqrt{2} \cdot e^{\frac{i\pi}{4}})^{2018}\right) = 2^{1009}.$$

Dominic: note that in both parts, the respective authors find *slightly more* than what they were required to. That is, respectively, the interpretation of k , and a bound on m_{2k+1} . The latter is an excellent example of the general notion that sometimes it is better to use a stronger statement than what you actually require in an induction argument (here for existence). The stronger statement (which you guess from playing with examples) makes the inductive step easier, as it's then clear that the new term you get is *distinct* from the terms you already have.

Problem 2

Parsing this question deserve at least a moment. Straight after a first reading, I find it worth writing down any key quantifiers which I might forget later. Here, it's the words *at most*. If you want to show the statement holds for $n = 2$, you need to investigate monic polynomials with degree zero, one and two. You should also make sure that any instances of x_i really are always distinct.

This matters in competitions! Two of our contestants failed to get the mark for showing $n = 2$ works, precisely because of not checking the linear case, and a third could have lost it for using examples which are sometimes not distinct. On hard papers, one mark actually is the difference between triumph and frustration. And of course it matters outside competitions too, since small cases are exactly what your reader might examine first, to check they understand the problem posed, so it's not a good place for awkward errors.

I started by trying to show that it couldn't possibly happen that *every* polynomial with degree at most n had this property, for some combinatorial reason. For example, that if every set of distinct integers could only be a solution set for a small number of polynomials, then we would end up with not enough polynomials. But I couldn't make this work at all; every bound ended up heavily in the wrong direction.

The next natural question is, does a *typical* polynomial of degree at most n have this property? But choosing a typical polynomial is hard, so in fact I asked, do the simplest

polynomials of degree at most n have this property? I think the simplest polynomials of degree at most n are $\{1, x, x^2, \dots, x^n\}$. Under what circumstances does

$$x_1^m + \dots + x_k^m = x_{k+1}^m, \tag{1}$$

have solutions in distinct integers? Famously, when $k = 2$ and $m \geq 3$ this is a very very hard problem indeed. So the first point is that it though it might be useful to use Fermat's Last Theorem, it would be foolish to pursue a strategy which, if successful, would have a proof of FLT as a sub-problem. At least, it would be foolish if the aim was to finish this strategy within a few hours.

So my main comment on this question is meta-mathematical. If lots of attempts at general arguments don't work, there must be some special example that does it. And what properties do I want this special example to have? Maybe one might have thought of this from scratch, but my motivation came from (1) in the case $m = p - 1$. Then, by Fermat's Little Theorem, all the summands are equal to 1 or 0 modulo p . If $k > p$, then after discounting any uniform factors of p , we obtain a congruence equation which is, in informal terms,

$$(0 \text{ or } 1) + \dots + (0 \text{ or } 1) \equiv (0 \text{ or } 1).$$

This looks really promising because it's quite restrictive, but it's still just a bit annoying: there are quite a few solutions. But it does give us the right idea, which is to find a polynomial P for which $P(x) \equiv 1$ modulo n . The equation $1 + \dots + 1 \equiv 1$ modulo n has solutions only if the number of summands on the LHS is 1 modulo n . So in this context, this reduces to showing that P is, additionally, injective on the integers, ie that $P(x) = P(y)$ only when $x = y$.

It's a nice exercise to show the existence of polynomials which are constant modulo n , and a good problem to work out how to force injectivity. If a polynomial is increasing everywhere, then it is certainly injective, and so the problem ends up being slightly easier in the case where the degree is odd than when the degree is even, but this is a nice conclusion to a nice problem, so I'll save it for any interested readers to finish themselves.

Problem 3

By Neel Nanda

If $|X| = n$, there are 2^n possible subsets, so at first glance the answer could be a variety of things, from a linear to an exponential function of n , each of which would suggest a different approach. So the first step is to conjecture an answer, and by examining small cases it seems impossible to do better than $2n - 2$. There are several natural constructions for this bound, such as n subsets of size $n - 1$ and $n - 2$ subsets of size 1, so we guess this to be our answer (which later turn out to be right!).

From here, a solution is deceptively simple, though empirically the five full solutions in the contest show that it was by no means easy to find. We proceed by induction on the size of X , and want to show that any collection of subsets S has size at least $2n - 2$. By assumption all subcollections are not tight, so if the union of a subcollection is not the whole set X , then there is an element which appears in exactly one subset. This is a useful result, so we'd like to force a subcollection whose union is not the whole set X .

One way to guarantee that the union of a subcollection is not X is by taking the subcollection of all subsets not containing some element b . So there is some element c which appears in only one subset not containing b . If we choose b so that it's the element contained in the fewest subsets of S , c is in at least as many subsets of S , but in only one subset not containing b . This means that at most one subset containing b doesn't contain c . This is useful, because after removing at most 2 subsets (the coefficient of n in $2n - 2$, importantly!), we now have that every subset in S either contains both b and c or neither. This means that we can replace the pair (b, c) with a new element d , to get a new collection of subsets S' of a set X' , of size $n - 1$, so by induction $|S| \leq |S'| + 2 \leq 2n - 2$.

There is also the case where all subsets contain b , but we can create an equivalent collection of subsets of $X \setminus \{b\}$ by removing b from all subsets. So again by induction we are done.

Problem 4

This question is quite unusual for an olympiad of this kind, and I was initially skeptical, but then it grew on me. Ultimately, I was unsurprised that many contestants attacked entirely with coordinate calculations. If you use this strategy, you will definitely get there in the end, but you have to accept that you aren't allowed to make any mistakes. And because of the amount of symmetry in the configuration, even if you make a mistake, you might still get the required answer, and so not notice that you've made a mistake. But I decided I liked it because various levels of geometric insight either reduced or removed the nastier calculations.

Typically, one could gain geometric insight by carefully observing an accurate diagram, but an accurate parabola is hard to draw. However, even from a vague diagram, we might guess the key intermediate property of the configuration, which is that the line joining the other two points in the quadrilateral is parallel to the y -axis. This means that they have the same x -coordinate, and indeed this x -coordinate must in fact be the same for any parabola through A and B , so it is reasonable to guess that it is $\frac{x_A + x_B}{2}$, the mean of the x -coordinates of A and B .

Since you know this is the goal, it's not too bad to calculate the equations of the tangent lines directly, and demonstrate this algebraically. But I was determined to use the focus-

directrix definition of a parabola. Recall³ that a parabola may be defined as the locus of points which are the same distance from a fixed point P (the *focus*), and a fixed line ℓ (the *directrix*). Naturally, the distance to the line is perpendicular distance.

To ensure the form given in the statement where y is a quadratic function of x , in this setting the directrix should be parallel to the x -axis. To define the tangent to the parabola at A , let A' be the foot of the perpendicular from A onto ℓ , so $AA' = PA$. I claim that the tangent at A is given by the perpendicular bisector of $A'P$. Certainly this passes through A , and it is easy to convince yourself that it can't pass through any other point B on the parabola, since $BA' > PB$, as A' is on ℓ but is not the foot of the perpendicular from B to ℓ . This final observation is truly a lot more obvious if you're looking at a diagram.

We now want to finish geometrically too. In our quadrilateral, one diagonal is parallel to the y -axis, and it will suffice to show that the existence of an incircle implies that A and B must have the same y -coordinate. We have just shown A and B are the same (horizontal) distance from the other diagonal. So certainly if they have the same y -coordinate, then the quadrilateral is a kite, and the sums of opposite sides are equal, which is equivalent to the existence of an incircle. One could then finish by arguing that this ceases to be true if you move one of A and B in either direction, or by some short explicit calculation if such a perturbation argument leaves you ill at ease.

Problem 5

This is a fairly classic competition problem, and while in my opinion the statement isn't particularly fascinating, it's interesting that it admits such a wide range of approaches.

As ever, you need to start by playing around with the setup, and guessing that the answer is $2n - 2$, and not thinking 'it can't possibly be the same answer as Q3??' Then think about reasons why you couldn't do better than $2n - 2$. My very vague reason was that if you only use horizontal sticks, the answer is clearly $2n - 2$, and the same if you only use vertical sticks. But it feels like you can only make life harder for yourself if you try to use both directions of sticks in lots of places. Note that some sort of argument involving averaging over stick lengths is definitely doomed to fail unless it takes into account the Latin square nature of the location of holes! For example, if you were allowed to put all the holes in the first row, $m(A)$ would be $n - 1$.

Induction is tempting. That is, you remove some number of sticks, probably those corresponding to a given hole, to reduce the board to an $(n - 1) \times (n - 1)$ configuration. If you do this, you need to be clear about why you can remove what you want to remove

³Possibly from A-level, though I haven't checked whether the syllabus has changed since I was at school, nor whether this is constant across exam boards. So if you are at school, maybe you should replace 'Recall that' with 'It is an interesting definition that'...

(in particular, the number of sticks you want to remove), and whether it's qualitatively different if the hole in question lies on the border of the board. In all of these settings, you want to be careful about 1×1 sticks, which it's easy inadvertently to count as both horizontal and vertical. This is unlikely to affect the validity of any argument (just picking either an arbitrary or a canonical direction if it's 1×1 should be fine) but does make it much harder to *check* the validity.

Joe exhibited directly a construction of $2n - 2$ cells which must be covered by different sticks. This approach lives or dies by the quality of the written argument. It must look general, even though any diagram you draw must, almost by definition, correspond to some particular case. Alternatively, since the problem is set on a grid, the cells correspond naturally to edges of a bipartite graph, where classes correspond to rows and columns. The holes form a perfect matching on this bipartite graph. But, as Harvey observed, if you split the rows and columns in two, on either side of the relevant hole (or not in the $2 + 2$ cases where the hole is at the border), you have a $(2n - 2) + (2n - 2)$ bipartite graph, and a perfect matching here corresponds to a set of cells which must be covered by different sticks. This is an ingenious idea, and if you've recently met Hall's Marriage Theorem, which gives a verifiable criterion for the existence of such a perfect matching, there are few better uses of your next ten minutes than to check whether Hall's condition a) should hold; b) can be proven to hold in this setting.

Problem 6

I thought this problem was extremely hard. The official solution starts with a 'magic lemma', that isn't quite so magic if you then read how it's used. The overall claim is that PQ , RS and AC are concurrent (or parallel), and this is proved using the fact that the radical axis of the two circles with diameters PQ and RS also passes through this point of concurrency. Hunting for key properties of subsets of points in the diagram is an important skill in hard olympiad geometry, since it exactly reflects how problem-setters produce the problems. All the more so when there is lots of symmetry in the construction. But this is a hard example - there are a lot of potentially relevant subsets of the configuration.

When you're really stuck with how to get involved in a synthetic configuration, you might consider using coordinates. Some of the UK students have been reading some chapters of a book⁴ focusing on various analytic approaches, so James and I felt it was safer to make sure we knew what the best settings were, and how far we could take them.

You almost certainly want the intersection of PR and QS to be your origin. I wanted to set up the configuration using the language of vectors, referenced by (P, Q, R, S) . This

⁴*Euclidean Geometry in Mathematical Olympiads* by Evan Chen. I've only had my own copy for a couple of days, but my initial impression is very positive - it fills a gap in the literature in a style that's both comprehensive and readable.

was because $PQ \perp BO$ and so on, hence $\mathbf{b} \cdot (\mathbf{q} - \mathbf{p}) = 0$ and so on. An alternative is to use complex numbers, which makes this condition a bit more awkward, but is more promising for the conclusion. Concyclicity is not a natural property in vectors unless you can characterise the centre of the circle, but can be treated via cross-ratios in \mathbb{C} . You also have to decide whether to describe the collinearity of A , B and P by expressing $\mathbf{p} = \lambda_{\mathbf{p}}\mathbf{a} + (1 - \lambda_{\mathbf{p}})\mathbf{b}$, or via something more implicit. There definitely are not four degrees of freedom here, since specifying A certainly defines at most one valid set of (B, C, D) , so one is mindful we'll have to eliminate many variables later. We also have to account for fact that \mathbf{r} is a negative scalar multiple of \mathbf{p} , and it's not clear whether it's better to break symmetry immediately, or use this towards the end of a calculation.

The point of writing this was that if your initial thought was 'this looks promising via coordinate methods', then I guess I agree. But there's a difference between looking promising, and actually working, and there are lots of parameterisation options. It's certainly worth thinking very carefully about which to choose, and in this case, challenging though they were, the synthetic or synthetic-trigonometric methods probably were better.

Results, comments, thanks

The results⁵ of the UK team were:

	P1	P2	P3	P4	P5	P6	Σ	
Joe Benton	7	6	7	7	7	0	34	Gold Medal
Rosie Cates	7	0	0	3	7	0	17	Honourable Mention
Neel Nanda	7	7	2	7	7	0	30	Silver Medal
Thomas Read	7	0	0	7	7	0	21	Bronze Medal
Alexander Song	7	0	0	3	1	1	12	Honourable Mention
Harvey Yau	7	7	7	7	7	0	35	Gold Medal

The cutoffs for medals were 18, 24, and 32. In fact, no students scored 31, 33 or $\{36, \dots, 41\}$. As a result, Harvey was joint 2nd, and Joe joint 5th among the 111 total participants. Neel's score sequence was, understandably, matched by several students, and so the number of gold medals had to be noticeably smaller than the ideal ratio. This means that all the gold medallists (three Korean, one Ukrainian, one Chinese, and our two British boys) solved at least one of the highly challenging Q3 and Q6, which feels very appropriate. Only the Chinese contestant scored full marks, which deserves everyone's admiration.

At this competition, team scores are given by the sum of the top three contestants. Following a slight change to the regulations, some countries brought four students, while others brought exactly six students, and others specified a subset of six students whose scores

⁵Full results for all teams can be found at http://rmms.lbi.ro/rmm2017/index.php?id=results_math

were in contention for the team score, and extra students who were only eligible for the individual contest. It should be remarked that Korea managed their remarkable score with two fewer students! The leading team scores were:

Korea (102), UK (99), China (94), Russian Federation (90), Ukraine (82), USA (79), Italy and Hungary (76), Romania (67).

This is the second year in a row that UK has placed second. Obviously the team would have enjoyed the thrill of placing first, but this is nonetheless an excellent achievement, and everyone should be very pleased with themselves. Many of the other highly-placed countries have rather different education systems, in which olympiad mathematics features much more heavily in schools, from primary age onwards. For our British contestants to have elevated themselves to this level mostly through their own enterprise reflects hugely well on them and the corps of volunteers who help to enrich their education.

This is Joe Benton's third gold medal at this competition. Joe is a very driven person, but also modest, so I hope it doesn't embarrass him too much to emphasise briefly the magnitude of this achievement. Over the ten years of this contest, we believe only three other students from any countries have earned even two golds. Harvey will have the chance to join this latter club in 2018. It seems unlikely that this new record will be exceeded, but smart young people tend to aim high, and we hope that Joe, Harvey and all their current colleagues may soon enjoy some challenges on the other side of the fence, energising the next generations of students towards whatever lofty contest goals they might set themselves, and in mathematics more generally.



On which note, there are many people who deserve thanks for organising this competition. In particular, Iulia Manicea, who helped us with all our arrangements, including

our pastoral idiosyncrasies; the UK team's guides, especially Gabriel and Ion, who were conscientious and helpful to a fault; Ilya Bogdanov and the problem composers, whose efforts define the high mathematical quality of the competition; and Radu Gologan, who makes everything happen every year at this remarkable event.

As for the UK group, thanks as always are due to Bev Detoef and the UKMT office for establishing all arrangements. We should not forget the countless teachers and, of course, parents, who have enabled, supported (and tolerated!) our students' hard work and progress. The current generation's appetite for material is colossal, and I want to single out Gabriel Gendler and Richard Freeland for their recent help in sating this appetite.

James and MT ran the summer school where I and many of my friends first became seriously enthused, and it was a pleasure to attend this contest, and discuss maths, pedagogy, and many other topics with them. Finally, our contestants, who worked tirelessly before and during the trip on dozens of interesting problems, and were ideal representatives of the UK and our maths enrichment activities, as well as being nice people and good company.

UK Team Diary

Wednesday 22 February

Did you know that trains in Moldova use different width tracks to trains in Romania? Well, I didn't know either, but I found out at 1am today, as my *wagon lit* from Chişinău was painstakingly jacked up to allow the transfer from ex-Soviet gauge to Western gauge. Outside, a man in a smart uniform and epaulettes shouted loudly and continuously at a group of men in smart uniforms without epaulettes. When their task was done, four sets of border and custom checks remained before the opportunity for another visit to the samovar, and finally a chance to sleep.

All of which is to say that I have arrived at maths competitions in better mental shape than 6am today at *Gara de Nord*. The UK students have a more conventional itinerary, but their flight from Luton doesn't arrive until mid-afternoon. After my first Haifa 'winter', I'm craving pork and snow, and find both in the mountain town of Sinaia, an hour away by train in Transylvania. I also find a bear. The bear seems very scared.

I return in time to meet the UK students as well as James and MT. Some of our contestants are now into their fourth year of attending international competitions, and the labour of finding them fresh material resembles Hercules against the hydra, but some problems on combinatorial geometry with convexity seem to have kept everyone entertained on the flight. Dinner is at the Moxa campus of the University of Economics, and features chicken with one of two possible carbohydrates, as in fact do the next six meals. However, today

is Thomas's 18th birthday, and so his parents have arranged a delicious cake, which elicits considerably more enthusiasm. On the short walk back to our meeting, we notice it is possible both to buy fireworks and get a tattoo among other options, so Thomas is spoiled for choice about how to take advantage of his majority.

The team's activities remain a mystery to James and me though, as we have to join the other leaders for the first meeting, to receive the proposed problems. We spend some time thinking about them separately then together, and our initial impression is that it's a very suitable paper, that hopefully our team will enjoy.

Thursday 23 February

The leaders meet to finalise the choice and statement of the problems. With a bit more time this morning, I've solved Q1, Q2, Q5, and proved Q3 once I'd looked up the correct bound. James eats conics for breakfast and shows me a glorious range of interpretations of Q4. We feel happy that our students will have a chance at all of these, while Q6 may prove more restricting. Either way, it's clearly an appropriate set for this competition, and is approved quickly. So it's time to finalise the English version of the paper, or finalize the American version. Many alternatives to the word *sieve* are proposed. Andrea from Italy is clearly already craving home comforts, but his suggestion of *cheese grater* is not taken up. This time I'm sorting the L^AT_EX, so get to settle the commas, but also take the blame for inconsistently spacing the rubric between the two papers. I'm sure everyone noticed.

While all this has been happening, the students have been at a lecture by Sergiu Moroianu at the Institute of Mathematics.

Joe Benton is the UK's roving reporter:

“The talk started with the observation that some algebraic identities have nice geometric analogues. For example, the fact that multiplication is commutative can be associated with the fact that we can calculate the area of a rectangle as the product of the two sides in either order. Similarly, the fact that multiplication is associative corresponds to the fact that we can calculate the volume of a cuboid by multiplying the three edge lengths together in any order. Then the lecturer investigated some of the properties of matrices in a similar manner, defining $SL_2(\mathbb{R})$ and linking the identity

$$\det(AB) = \det(A)\det(B)$$

with the geometrical interpretation of $\det(A)$ as an (area) scale factor.

“In order to gain further algebraic insights, the next idea was to try investigating different geometries, and to do this the different geometries had to be classified. This led on to a discussion of the axioms of Euclidean geometry and, more specifically, the independence

of the parallel postulate from Euclid's other axioms. To show this the lecturer exhibited a model of the hyperbolic plane, taking the points to be the complex numbers z such that $\text{Im}(z) > 0$ and lines to be vertical lines in the complex plane or half-circles with diameter on the real line.

“It then turns out that there is a distance function for this model that satisfies all the usual axioms, as well as a set of isometries of the hyperbolic plane that are similar to Möbius transformations and allow us to compute the area of a given triangle in the hyperbolic plane as $\pi - a - b - c$, where a, b, c are the angles of the triangle. Finally, an extension of the hyperbolic plane to three dimensions so that we could study the area of tetrahedra allowed us to gain an algebraic equivalence relating the complex logarithm and argument functions by counting the volume formed by five points in two different ways.”

For all the charms of Chipping Norton, I sense MT is enjoying the grittier nature of Bucharest Sector 1, and has been shepherding the students round various sites in between attempts at practice problems. I join them for a brief visit to a geology museum. I am very cynical, but it slightly exceeds my expectations, and is infinitely better than the nearby Museum of the Romanian Peasant, which currently ties with the Hanoi Ethnology Museum as my least favourite olympiad excursion of all time.

The opening ceremony is held in the grand hall of the university, and includes several welcoming and thoughtful speeches from the Mayor of Bucharest and the headteacher of Tudor Vianu, the school which hosts this competition every year. Each team briefly presents themselves on stage. Joe and Neel have accumulated a large collection of UK flags from previous competitions, and should hereby consider themselves publicly shamed for forgetting their promise to bring them. It is over soon, and while the students enjoy a quiet evening and an early night, the leaders have to finalise markschemes for all the problems. The walk back takes us through Victory Square, and past the protesters whose fires and slogans have been on front pages around the world in the past months. It's an interesting time, and the atmosphere of this city feels very different from my first visit, for the inaugural edition of this competition in 2008.

Friday 24 February

The first day of the contest starts at 9am. The British students seem fairly relaxed, and hopefully are aiming high. Contestants may ask questions of clarification during the first 30 minutes. Rosie does this, and I send my reply to her two queries back via the courier. Five minutes later it is returned to me with the explanation that the student does not understand the answer. Even under competition pressure this seems unlikely, given that my answers are, respectively ‘yes’, and putting a ring around one of three options she has listed. It turns out that actually the student courier did not understand what to do with the answer, and the situation is quickly corrected.

We approve more markschemes. The US deputy leader Po-Shen and I share our views on the challenge of correctly finding the bound in Q3, and our suggestion that this instead be worth 2 points is upheld. Various further discussions fill the morning, and we return just in time to meet the students at the end of the exam. Harvey claims all three problems with a relaxed grin, while Joe claims all three problems with the haunted look of a man whose twelfth espresso of the day has just worn off. Alexander and Thomas say that they spent most of the time making sure their solutions to Q1 were totally watertight, which, given the intricacy of the arguments, was clearly a very sensible strategy.

To provide a distraction, if not actually a break from time-pressured problem-solving, I've booked a pair of escape rooms for the UK students later in the afternoon. Bucharest is the home of these games, where the aim is to solve themed puzzles as part of a story in time to escape a locked room. I join one of the rooms, where there are some theatrical reveals involving wrenches, and clues hidden in combination-locked cabinets, where ability to add three-digit numbers proves useful. Neel's carrying voice means we get to enjoy some of the drama and twists of the other room too. Anyway, this proved an ideal way to avoid useless post-mortems, and I highly recommend Vlad and his pair of rooms⁶.

Later, James and I get to look at the students' work from this morning. Their assessments are pretty accurate. Harvey's solutions to everything are beautiful, while Neel's bounding argument in Q2 is certainly the most vulgar (and, in fact, unnecessary) calculation of the year so far. Joe's solution to Q3 bears such obvious resemblance to an official solution that his uncharacteristic abundance of small errors probably won't matter, including the memorable set $A_i \setminus \{i\}$, where the two i s mean different things. Some of the team might reflect that a moment of casualness in checking the $n = 2$ case on Q2 is a frustrating way to lose a potential mark, but when I compare notes with James, it sounds like the slow and steady approach to Q1 has indeed paid off for everyone, so hopefully it will not be too painful to agree the scores tomorrow.

Saturday 25 February

It's the second day of the competition, and the UK team look bright-eyed and positive at breakfast. They aren't the only ones under pressure this morning, as James and I must settle the scores from yesterday's questions with local markers, known as coordinators. It's hard to guess in how much detail one will have to explain your contestants' scripts, so it is safer to prepare almost line-by-line. On this occasion though, perhaps we have over-prepared, as every meeting ends quickly with offers of 7/7 exactly where we were hoping, and indeed in a couple of places where we were not hoping. The markschemes are very clear about certain omissions which carry a point deduction, so to ensure fairness

⁶<http://www.gameoftrolls.ro/?lang=en>

and consistency, we insist that two scores are moved down. I'm confident that any British student would prefer an honourable 41/42 than an accidental 42/42.

No-one's going to be scoring 41 nor 42 unless they solve the extremely challenging geometry Q6, and as we meet our students afterwards, it turns out they have not managed any progress there. However, they claim an almost full set of solutions to Questions 4 and 5, which, if accurate, is a very good return. Everyone is in a good mood, and after I explain a couple of approaches to Q6, no-one seems too disappointed that they didn't spot these.

There are various schedules floating around, listing multiple locations and times for lunch, but our space-time trajectory intersects none of them, so we follow the Chinese team to a recommended cheap Szechuan restaurant round the corner. Various circle theorems are explored via the Lazy Susan, and there is a grand reveal of the marks we've recently confirmed. There's time for another pair of escape rooms while the second day scripts arrive. As Rosie remarks, two in two days can lead to excessive outside-the-box thinking. Sometimes a radiator really isn't a sinister prop, a device for encoding five-digit numbers, or a clue to a Templar tunnel; it's just a radiator. Otherwise we'd be cold.

When the scripts arrive, as expected the cupboard is pretty bare on Q6. If there were marks for quantity, Neel might get some, and if there were marks for most uses of esoteric theory in a single page, Alexander might get one. No set of scripts for an international-level medium combinatorics problem will ever be perfect, but our Q5s come close. It's therefore not a long evening, and we can join the students for dinner with the American team. For most of them it's their first visit to Europe, and there's much comparing of culture and maths training programmes. There's also a long discussion of whether it's sensible to teach maths in primary school. Those present who have small children or younger siblings weigh in on the mysteries of the 'grid method', and whether toddlers implicitly understand commutativity, even if they can't spell it.

Sunday 26 February

The UK leaders gather early in the 'philosophical anti-cafe' opposite Vianu school, to ponder the final scripts with a coffee and a view of an artfully-arranged folio of Spinoza. James has a loyalty card here. Unfortunately two of our students have clear algebraic errors in Q4, but apart from that everything is very straightforward. Though following last night's conversation, we note that maybe a revision clinic on mathematical spelling might prove useful. Anonymous student *X* thinks there's one L in 'ellipse', counterbalanced by anonymous student *Y* who thinks there are two in 'column'. The word 'parallel' comes in many disguises.

Of course, the coordinators couldn't care less about that, and they don't even mind Neel's two-cases-at-once inductive step, so again we get what we ask for on Q5 immediately, and

on Q4 in the time it takes James to draw a lozenge tiling representing Thomas's shearing argument. For Q6, it turns out there clearly *is* a mark for most uses of esoteric theory in a single page, so Alexander gets it. They show us a diagram with over a hundred lines which suggests that the exotic equivalence he claims is actually true. There we go. Overall, the quality of our written solutions has been extremely high. It feels like I say this every time now, but it isn't idle propaganda. We remember the horrors that used to emerge occasionally, and the effort to make this improvement permanent feels well worth it.

Meanwhile, to fill the day, the students have gone to Sinaia. Two of their guides went with them to help with tickets at the station, apparently under the impression that never having taken a train before wouldn't be an obstacle to this role. Either way, they made it, and following my request for material for this report, I receive a trickle of presentable photos, though there is talk afterwards of some rather more informal versions which are apparently not suitable. The Transylvanian winter is thawing, but slowly and messily, and Harvey reports that several of the group spent more time horizontal than vertical. Irrespective of their preferred axis, there's no comment on whether they saw my bear, or any other bear. But since my bear was scared of me, one wonders what it would make of MT's telling-off face? (Last seen by me during the notorious 'balcony incident' at a summer school in 2005, but hardly forgotten.)

The students return in time for confirmation of the results and their medals. As so often, there is pleasure that we have done so well collectively, mixed with mild disappointment for those who ended up just short of a boundary, and that the UK was so close to placing first. Because of the strength of the invited countries, earning a medal of any colour is a very worthwhile achievement, and so Rosie is impressively sanguine about missing out so narrowly in such an unfortunate manner. Alexander was closer than it appears, and could have two more opportunities to take part.

The closing ceremony at Vianu school proceeds rapidly. There is the usual challenge of photographing the students receiving their prizes, but this time is easy. Thomas is about a foot taller than everyone else on the stage, while Neel is flanked by almost the entire Russian team, but his chutzpah trumps their numerical advantage, with laughter all round. Joe claims this year's gold medal is substantially weightier. He hasn't brought his previous pair, so the chance to verify this and recreate a Mark Spitz moment goes begging.

It's 7pm, and UK student enthusiasm for the closing disco (not my words) is about as high as MT's enthusiasm to chaperone the closing disco. Instead we find a Middle Eastern restaurant, and it's refreshing to eat hummus in a place which doesn't claim to be the 'best in Israel' though I don't think Abu Said in Akko will be rushing to steal the recipe. Po-Shen outlines his vision of a year-long maths camp. I think present company are tired enough after five days here. Some are interested to view, if not actually participate in, the protests in Victory Square, but it seems tonight is a quiet one and nothing is being burned, so late-night cards and a perusal of each others' scripts will have to do.

Monday 27th February

The rest of the group have a flight back to London later today which apparently cost 99p per person before tax. I don't know how much less the 5am option was, but I think it's probably worth it. My own flight is truly at 5am tomorrow and I plan to stay up all night. The students return to school tomorrow, doubtless to receive a glorious mix of adulation and apathy. Harvey requests whether next year this trip can be timed differently so that he can miss the whole of his local Eisteddfod, rather than just one day. I promise to ask the organisers, say goodbye, then head for the hills on a train journey long enough to write the entirety of this report.

3am at Bucharest airport, and thoughts can now turn to the future. Many of us will meet in five weeks' for another round of mathematics in the more tranquil setting of Cambridge. Meanwhile, I certainly enjoyed, admittedly through red eyes, the entertainment of a flight to Israel where baggage size regulations are actually enforced at the boarding gate, and apparently everyone else made it back safely too.

